QUADRATIC FORMS AND CHAIN SEQUENCES

ROBERT SEALL AND MARION WETZEL

It has been shown that a monotone Hausdorff moment sequence can be characterized by Stieltjes type quadratic forms. This was a result of the theory of chain sequences [4]. In the present paper we shall characterize an "extended" monotone Hausdorff moment sequence by Jacobi type quadratic forms. This will be done by extending the properties of ordinary chain sequences.

The paper is in two sections. In the first section a relation between chain sequences and Jacobi type quadratic forms will be established. The connection between the "extended" monotone Hausdorff moment problem and chain sequences will be discussed in the second section.

1. Jacobi forms and chain sequences. In the theory of positive definite continued fractions a sequence of nonnegative Stieltjes forms is characterized. It is shown that the quadratic form

$$\sum_{p=1}^{n} x_p^2 - 2\sum_{p=1}^{n-1} a_p x_p x_{p+1}, \qquad n = 2, 3, 4, \cdots, \tag{1.1}$$

is positive semidefinite if and only if

$$a_p^2 = (1 - g_{p-1})g_p,\ 0 \leqq g_{p-1} \leqq 1, \qquad p = 1, 2, 3, \cdots \quad (g_0 = 0). \tag{1.2}$$

This result is an immediate consequence of a more general theorem characterizing positive definite Jacobi fractions.

We shall now state an analogous theorem for Jacobi forms.

THEOREM 1.1. *The quadratic forms*

$$\sum_{p=1}^{n} (1 + b_p)x_p^2 - 2\sum_{p=1}^{n-1} a_p x_p x_{p+1}, \tag{1.3}$$

and

$$\sum_{p=1}^{n} (1 - b_p)x_p^2 - 2\sum_{p=1}^{n-1} a_p x_p x_{p+1}, \qquad n = 2, 3, 4, \cdots, \tag{1.4}$$

are positive semidefinite if and only if

$$a_p^2 = 4(1 - g_{2p-2})(1 - g_{2p-1})g_{2p-1}g_{2p}, \tag{1.5}$$

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$$(1.6)\qquad \begin{aligned} b_p &= 1 - 2(1 - g_{2p-3})g_{2p-2} - 2(1 - g_{2p-2})g_{2p-1}, \\ &= 2(1 - g_{2p-2})(1 - g_{2p-1}) + 2g_{2p-3}g_{2p-2} - 1, \end{aligned}$$

where

$$0 \leqq g_{p-1} \leqq 1, \qquad p = 1, 2, 3, \cdots \quad (g_{-1} = 1, g_0 = 0).$$

Such sequences shall be called double chain sequences. The a_p^2 and b_p shall be referred to as the elements and the g_p as the parameters. Double chain sequences arise in the continued fraction solution of the "extended" monotone Hausdorff moment problem [3].

Proof. We designate the determinants associated with the quadratic forms (1.3) and (1.4) by $B_p(1)$ and $B_p(-1)$. Their recurrence formulas are

$$(1.7)\qquad B_{p+1}(1) = (1 + b_{p+1})B_p(1) - a_p^2 B_{p-1}(1)$$

and

$$(1.8)\qquad \begin{aligned} B_{p+1}(-1) = (1 - b_{p+1})B_p(-1) - a_p^2 B_{p-1}(-1), \quad p = 0, 1, 2, \cdots \\ (B_{-1}(1) = B_{-1}(-1) = 0,\ B_0(1) = B_0(-1) = 1). \end{aligned}$$

It can be shown that if the a_p^2 and b_p are of the forms (1.5) and (1.6), then

$$(1.9)\qquad B_p(1) = 2^p(1 - g_1)(1 - g_2) \cdots (1 - g_{2p-1})$$

and

$$(1.10)\qquad \begin{aligned} B_p(-1) = 2^p g_1(1 - g_2)g_3 \cdots (1 - g_{2p-2})g_{2p-1}, \\ p = 1, 2, 3, \cdots \quad (g_0 = 0), \end{aligned}$$

where the g_p are not necessarily between zero and one. These relations follow immediately by an induction on (1.7) and (1.8).

The proof consists of two cases.

Case 1. The a_p^2 all positive. We consider the necessity part of the proof. The nonnegativity of the forms (1.3) and (1.4) implies that $B_p(1) \geqq 0$ and $B_p(-1) \geqq 0$. We observe moreover that the $B_p(1)$ and $B_p(-1)$ are positive by (1.7) and (1.8). Set $b_1 = 1 - 2g_1$. From (1.9) and (1.10) it follows that $0 < g_1 < 1$. Next write

$$(1.11)\qquad a_1^2 = 4(1 - g_1)g_1g_2$$

and

$$(1.12)\qquad b_2 = 1 - 2(1 - g_1)g_2 - 2(1 - g_2)g_3.$$

Notice that $g_2 \neq 1$. The formulas (1.9) and (1.10) give

$$B_2(1) = 2(1 - g_2)(1 - g_3)B_1(1) \tag{1.13}$$

and

$$B_2(-1) = 2(1 - g_2)g_3B_1(-1). \tag{1.14}$$

From (1.11), (1.13) and (1.14) we see that g_2, $(1-g_2)(1-g_3)$ and $(1-g_2)g_3$ are each positive. It follows that $0 < g_2 < 1$ and $0 < g_3 < 1$.

We proceed with the induction. Suppose that $0 < g_{2p-1} < 1$ for $p = 1, 2, 3, \cdots, k$, $k \geqq 2$. Set

$$a_k^2 = 4(1 - g_{2k-2})(1 - g_{2k-1})g_{2k-1}g_{2k}, \tag{1.15}$$

and

$$b_{k+1} = 1 - 2(1 - g_{2k-1})g_{2k} - 2(1 - g_{2k})g_{2k+1}. \tag{1.16}$$

It is necessary to show that g_{2k} is not equal to one. The following argument is given. Since $B_{k+1}(1) > 0$, using (1.9), and the definition of a_k^2, relation (1.7) yields the inequality $(1 + b_{k+1}) > 2g_{2k-1}g_{2k}$. In a similar manner using (1.8) and (1.10), we obtain

$$(1 - b_{k+1}) > 2(1 - g_{2k-1})g_{2k}.$$

Adding these inequalities we have $g_{2k} < 1$. By (1.9) and (1.10) we have

$$B_{k+1}(1) = 2(1 - g_{2k})(1 - g_{2k+1})B_k(1) \tag{1.17}$$

and

$$B_{k+1}(-1) = 2(1 - g_{2k})g_{2k+1}B_k(-1). \tag{1.18}$$

From (1.15), (1.17), (1.18) and our assumptions under the induction it follows that g_{2k}, $(1-g_{2k})(1-g_{2k+1})$, and $(1-g_{2k})g_{2k+1}$ are positive. Hence $0 < g_{2k} < 1$ and $0 < g_{2k+1} < 1$, and the induction is complete.

The sufficiency part of the proof follows immediately from (1.9) and (1.10). It is important to notice that the formulas (1.9) and (1.10) hold only with $g_0 = 0$. The existence of such a sequence of minimal parameters will be discussed in §2.

Case 2. The a_p^2 nonnegative. In this case the determinants $B_p(1)$ and $B_p(-1)$ separate into blocks of nonoverlapping determinants. We suppose that $a_1^2, a_2^2, \cdots, a_{p-1}^2$ are positive and a_p^2 is zero. (This corresponds to the case of a terminating continued fraction.) It will be seen that the first p forms in (1.3) and (1.4) are positive semi-definite if and only if $a_1^2, a_2^2, \cdots, a_{p-1}^2$ and $b_1, b_2, \cdots, b_p$ form a terminating double chain sequence. The proof is similar to Case 1.

If a finite number of the a_p^2 are equal to zero, we simply repeat the argument in Case 2 for each finite block and the argument in Case 1 for the infinite block. If an infinite number of the a_p^2 are equal to zero, we repeat the argument of Case 2 for each finite block. This completes the proof.

We remark that Theorem 1.1 can be obtained by an inductive process using the theorem indicated at the beginning of the section [4]. This theorem says that the quadratic form

$$\sum_{p=1}^{n} \beta_p x_p^2 - 2 \sum_{p=1}^{n-1} \alpha_p x_p x_{p+1}, \qquad n = 2, 3, 4, \cdots, \tag{1.19}$$

is nonnegative if and only if $\beta_n \geqq 0$ and there exist numbers $h_0, h_1, \cdots,$ such that $\alpha_n^2 = \beta_n \beta_{n+1}(1-h_{n-1})h_n$, $0 \leqq h_{n-1} \leqq 1$, $n=1, 2, 3, \cdots$. In the proof of Theorem 1.1 by this method we apply the transformation

$$h_n = \frac{g_{2n-1}g_{2n}}{g_{2n-1}g_{2n} + (1-g_{2n})(1-g_{2n+1})}, \quad n = 1,2,3, \cdots \quad (h_0 = 0), \tag{1.20}$$

in (1.3) and

$$h_n = \frac{(1-g_{2n-1})g_{2n}}{(1-g_{2n-1})g_{2n} + (1-g_{2n})g_{2n+1}}, \quad n = 1,2,3, \cdots \quad (h_0 = 0), \tag{1.21}$$

in (1.4).

2. **Jacobi fractions and moment sequences.** In the theory of totally monotone sequences the monotone Hausdorff moment problem is solved [5]. A sequence of real numbers $\{c_n\}$, $n=0, 1, 2, \cdots$ $(c_0=1)$, is said to be a monotone Hausdorff moment sequence if there exists a monotone nondecreasing real function $\phi(u)$, $0 \leqq u \leqq 1$, such that $c_n = \int_0^1 u^n d\phi(u)$, $n=0, 1, 2, \cdots$. It is shown that such a sequence is a monotone Hausdorff moment sequence if and only if the power series $\sum_{n=0}^{\infty} c_n x^n$ has a Stieltjes type continued fraction expansion of the form

$$\frac{1}{1} - \frac{(1-g_0)g_1 x}{1} - \frac{(1-g_1)g_2 x}{1} - \cdots, \tag{2.1}$$

where $0 \leqq g_n \leqq 1$, $n=0, 1, 2, \cdots$.

The continued fraction solution of the "extended" monotone Hausdorff moment problem has recently been given [3]. A sequence of real numbers $\{c_n\}$, $n=0, 1, 2, \cdots$ $(c_0=1)$, is said to be an "extended" monotone Hausdorff moment sequence if there exists a monotone nondecreasing real function $\phi(u)$, $-1 \leqq u \leqq 1$, such that $c_n = \int_{-1}^{1} u^n d\phi(u)$, $n=0, 1, 2, \cdots$.

THEOREM 2.1. *The sequence* $\{c_n\}$ *is an "extended" Hausdorff moment sequence if and only if the power series* $\sum_{n=0}^{\infty} c_n x^n$ *has a Jacobi type continued fraction expansion*

$$\frac{1}{1+b_1x-}\ \frac{a_1x^2}{1+b_2x-}\ \frac{a_2x^2}{1+b_3x-}\ \cdots, \tag{2.2}$$

in which $\{a_n\}$, $\{b_n\}$ *form a double chain sequence.*

We shall indicate how the theorem may be obtained by a transformation. It is well known that $P(z)=\sum_{n=0}^{\infty} c_n z^n$ is a moment generating function for the "extended" monotone Hausdorff moment problem if and only if $Q(w)=(1+z)P(z)$, where $w=(2z/(1+z))$, is a moment generating function for the regular monotone Hausdorff moment problem.

PROOF. By replacing x by $2x/(1+x)$ in the even part of (2.1) and making an equivalence transformation we obtain a continued fraction of the type (2.2) in which the a_p and b_p are of the forms (1.5) and (1.6). Using the theorem stated at the beginning of the section the proof is immediate.

It is possible to extend several further results in the theory of ordinary chain sequences. One of these pertains to the minimal parameters of a chain sequence. In the proof of Theorem 1.1 the following result was used: A double chain sequence $\{a_p^2\}$, $\{b_p\}$ has minimal parameters l_p, m_p, where $0\leqq l_p\leqq g_{2p-1}$, $0\leqq m_p\leqq g_{2p}$, $p=0, 1, 2, \cdots$, given by

$$l_1=\frac{1-b_1}{2},$$

$$l_{p+1}=\begin{cases}0 \quad \text{if} \quad m_p=1,\\ \dfrac{b_{p+1}-[1-2m_p(1-l_p)]}{-2(1-m_p)} \quad \text{if} \quad m_p<1,\end{cases} \tag{2.3}$$

$$p=1, 2, 3, \cdots,$$

$$m_0=0,$$

$$m_{p+1}=\begin{cases}0 \quad \text{if} \quad m_p=1, \quad \text{or} \quad l_{p+1}=0 \text{ or } 1,\\ \dfrac{a_{p+1}^2}{4(1-m_p)l_{p+1}(1-l_{p+1})} \quad \text{if} \quad m_p<1, 0<l_{p+1}<1,\end{cases} \tag{2.4}$$

$$p=0, 1, 2, \cdots.$$

The proof of (2.3) and (2.4) follows by the methods of ordinary chain sequences.

The minimal parameters of a double chain sequence have been shown to have a geometric representation in "extended" Hausdorff moment spaces [1], [2], [3].

By contractions and the methods for ordinary chain sequences expressions for the maximal parameters of a double chain sequence can be obtained. In addition to these formulas, theorems (11.2), (19.1), and (20.2) of [4] can also be extended by the same methods to the case of double chain sequences.

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Illinois Institute of Technology and
Denison University