$$\int_C |u(x) - u_C|\, dx \leqq K_3 R^{n+1-n/p}\left(\int_C |\operatorname{grad} u(x)|^p\, dx\right)^{1/p} \tag{17}$$

and the desired result follows from the previous theorem.

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University of Minnesota

PHRAGMÉN-LINDELÖF THEOREMS FOR SECOND ORDER QUASI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

JOHN O. HERZOG[1]

Phragmén-Lindelöf theorems for uniformly elliptic partial differential equations have been the subject of several papers in recent years (see e.g. [**3**; **4**; **6**; **7**; **8**; **10**]). Here we are concerned with the Phragmén-Lindelöf theorem for second order quasi-linear elliptic equations of the form

$$L[z] \equiv \sum a_{ij}(x, p) z_{x_i x_j} = f(x, z, p), \tag{1}$$

which *need not* be uniformly elliptic. The main result is Theorem **1** which roughly says that if $u(x)$ is a subfunction with respect to (1) in a domain D contained in a half space and if $u(x) \leqq 0$ on the boundary of D then either $u(x) \leqq 0$ throughout D or the maximum of $u(x)$ on a sphere of radius r is of order not less than r^η for some $\eta > 0$. Probably the most interesting feature of this theorem is that its proof essentially depends only on the behavior of the functions $a_{ij}(x, p)$ and $f(x, z, p)$ for $\sum p_i^2 \leqq 1$. For $f \equiv 0$ and dimension $n = 2$ it is shown that $\eta = 1$.

Let D be an unbounded domain contained in a half space of n-dimensional Euclidean space and let T be the domain in $2n$-dimen-

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sional Euclidean space defined by $T\equiv\{(x, p)\equiv(x_1, \cdots, x_n, p_1, \cdots, p_n): x\in D$ and $-\infty<p_1, \cdots, p_n<+\infty\}$. Throughout the paper we shall use the notation: $p_i=\partial z/\partial x_i$, $r_{ij}=\partial^2 z/\partial x_i\partial x_j$, $\|x\|^2=\sum_1^n x_i^2$. $\sum$ will denote summation over one or both of the indices i, j, $1\leqq i, j\leqq n$.

Equation (1) is considered subject, at various times, to certain of the following conditions:

(i) The $a_{ij}(x, p)$ are continuous and have continuous first partial derivatives with respect to the p_i for all $(x, p)\in T$.

(ii) $\sum a_{ij}(x, p)\mu_i\mu_j>0$ for all real $\mu_1, \cdots, \mu_n$ (not all zero) and all $(x, p)\in T$ (i.e. we assume pointwise ellipticity).

(iii) $a_{ij}=a_{ji}$ and the determinant of the (a_{ij}) is identically one on T.

(iv) There exists an $\alpha_0>0$ such that $(\sum a_{ii})^n\leqq\alpha_0$ for all $(x, p)\in T$ with $\sum p_i^2\leqq 1$.

(v) $f(x, z, p)$ is continuous and has continuous first partial derivatives with respect to $z, p_1, \cdots, p_n$, with $f_z\geqq 0$ for all $x\in D$ and for all z and p_i.

(vi) There is a fixed $m>n^{1/2}$ such that there exist constants β and γ so that $|f(x, z, p)|\leqq\beta(p_1^2+\cdots+p_n^2)^\gamma$ for all $x\in D$ whenever $\sum p_i^2\leqq 1$, where $\gamma\geqq(2-\eta)/(2-2\eta)$, and

$$\eta=\log_3\frac{1-\exp[-8m\alpha_0/(n-1)^{n-1}]}{1-\exp[-2m\alpha_0/(n-1)^{n-1}]}.$$

The proofs will be based on the following principle:

MAXIMUM PRINCIPLE [5]. Let D be a bounded domain in Euclidean n-space and let $F(x_1, \cdots, x_n, z, p_1, \cdots, p_n, r_{11}, r_{12}, \cdots, r_{nn})$ be continuous and have continuous first partial derivatives with respect to $z, p_1, \cdots, p_n, r_{11}, r_{12}, \cdots, r_{n-1,n}$ and r_{nn} for all $x\in D$ and all $(z, p_1, \cdots, r_{nn})$, $-\infty<z, p_1, \cdots, p_n, r_{11}, r_{12}, \cdots, r_{nn}<\infty$. Furthermore, assume that $F_z\leqq 0$ and that the quadratic form $\sum F_{r_{ij}}\mu_i\mu_j$ is positive definite for all $x\in D$ and all $(z, p_1, \cdots, r_{nn})$. Then if $z_1(x)$ and $z_2(x)\in C^{(2)}(D)$, if the upper limit of $z_1(x)$ is less than or equal to the lower limit of $z_2(x)$ as x approaches any boundary point of D, and if

$$F(x_1, \cdots, x_n, z_1, z_{1x_1}\cdots, z_{1x_n}, z_{1x_1x_1}, z_{1x_1x_2}, \cdots, z_{1x_nx_n})\geqq 0$$

in D and

$$F(x_1, \cdots, x_n, z_2, z_{2x_1}, \cdots, z_{2x_n}, z_{2x_1x_1}, z_{2x_1x_2}, \cdots, z_{2x_nx_n})\leqq 0$$

in D, it follows that either $z_1<z_2$ in D or $z_1\equiv z_2$ in D.

The conditions placed on equation (1) are such that $L[z]-f(x, z, p)$ satisfies the hypothesis placed on F in the maximum principle.

THEOREM 1. *Assume equation* (1) *satisfies conditions* (i)-(vi). *Let* $u(x)$ *satisfy in* D *the inequality* $L[u] \geqq f(x, u, p)$ *and let the upper limit of* $u(x)$ *be nonpositive as* x *approaches any point on the boundary of* D. *Then if*

$$\limsup_{r\to\infty} \frac{M(r)}{r^\eta} \leqq 0,$$

where $M(r) = \text{l.u.b. } u(x)$ *for* $\|x\| = r$, $x \in D$, *it follows that* $u(x) \leqq 0$ *throughout* D.

PROOF. In addition to the previously defined notation we let

$$t^2 = x_1^2 + \cdots + x_{n-1}^2, \qquad G_R^i = D \cap \{x : t^2 + (x_n + R)^2 \leqq (iR)^2\},$$

$$S_R^i = \{x : t^2 + (x_n + R)^2 = (iR)^2,\ x_n \geqq 0\},$$

$$N_i(R) = \text{l.u.b. } u(x) \text{ for } x \in S_R^i \cap D, \qquad i = 2, 4.$$

$$D_{iR} = \{x : x \in D \text{ and } \|x\| \leqq iR\}, \qquad i = 1, 2, 4.$$

Without loss of generality consider D contained in the half space $x_n > 0$. Assume $\lim \sup_{r\to\infty} M(r)/r^\eta \leqq 0$ and that $M(R') > 0$ for some R' (and hence by the maximum principle for all $R \geqq R'$). For $R \geqq R'$ define

$$w_R(x) = \frac{t^2 + (x_n + R)^2}{4R^2}.$$

The following preliminary bounds will be used to show that there exists a constant $\alpha > 0$, dependent on α_0, m, and n, such that

$$W_R(x) \equiv N_2(R) \frac{1 - e^{-\alpha w_R}}{1 - e^{-\alpha}}$$

satisfies $L[W_R] \leqq f(x, W_R, W_{Rx})$ for all $x \in G_R^4$ when R is sufficiently large. Note that by the maximum principle, $N_2(R) > 0$.

By the Schwarz inequality,

$$\left| \sum a_{ij} w_{Rx_ix_j} \right| \leqq \left(\sum a_{ij}^2\right)^{1/2} \left(\sum w_{Rx_ix_j}^2\right)^{1/2}, \tag{2}$$

and, since the quadratic form is assumed positive definite, we have [2, p. 32]

$$\sum a_{ij} w_{x_i} w_{x_j} \geqq \left[\min_{1 \leqq i \leqq n} \lambda_i(x, p)\right] \sum w_{x_i}^2,$$

where the $\lambda_i(x, p)$ are the characteristic roots of $A \equiv (a_{ij})$ at the point $(x, p) \in T$. Using a well-known inequality [1, p. 67], we have

$$\lambda_{\min}^{A} \geqq [\det A]\left[\frac{n-1}{\operatorname{Trace} A - \lambda_{\min}^{A}}\right]^{n-1} \geqq [\det A]\left[\frac{n-1}{\sum a_{ii}}\right]^{n-1},$$

where $\lambda_{\min}^{A}$ denotes a minimum characteristic root of the real positive definite matrix A. Hence, since $\det A = 1$,

$$(3) \qquad \sum a_{ij} w_{Rx_i} w_{Rx_j} \geqq \left[\frac{n-1}{\sum a_{ii}}\right]^{n-1} \sum w_{Rx_i}^{2}.$$

Also,

$$(4) \qquad \sum a_{ij}^{2} = \left(\sum a_{ii}\right)^{2} - \sum (a_{ii}a_{jj} - a_{ij}^{2}) \leqq \left(\sum a_{ii}\right)^{2}.$$

Thus by (2), (3), and (4) above we see that

$$(5) \qquad \begin{aligned} &L[W_R] - f(x, W_R, W_{Rx}) \\ &= N_2(R)\frac{\alpha e^{-\alpha w_R}}{1-e^{-\alpha}}\left\{\sum a_{ij} w_{Rx_ix_j} - \alpha \sum a_{ij} w_{Rx_i} w_{Rx_j}\right\} - f(x, W_R, W_{Rx}) \\ &\leqq \frac{(n-1)^{n-1} N_2(R) \alpha e^{-\alpha w_R} \sum w_{Rx_i}^{2}}{(1-e^{-\alpha})(\sum a_{ii})^{n-1}} \\ &\cdot \left\{ \frac{(\sum a_{ii})^{n} [\sum w_{Rx_ix_j}^{2}]^{1/2}}{(n-1)^{n-1}\sum w_{Rx_i}^{2}} - \alpha - \frac{(1-e^{-\alpha})(\sum a_{ii})^{n-1} f(x, W_R, W_{Rx})}{N_2(R)\alpha(n-1)^{n-1} e^{-\alpha w_R} \sum w_{Rx_i}^{2}} \right\} \\ &\equiv G \cdot H, \end{aligned}$$

where H denotes the expression contained in the braces.

Now,

$$(6) \qquad 1/4R^{2} \leqq \sum w_{Rx_i}^{2} \leqq 4/R^{2} \qquad \text{for } x \in G_R^{4}$$

and $w_{Rx_ix_i} = 1/2R^2$ while $w_{Rx_ix_j} = 0$ for $i \neq j$. Therefore

$$(7) \quad \left(\sum w_{Rx_ix_j}^{2}\right)^{1/2} / \sum w_{Rx_i}^{2} \leqq [(n/4R^{4})^{1/2}]/[1/4R^{2}] = 2n^{1/2} \text{ for } x \in G_R^{4}.$$

Since $w_R(x) \geqq 1/4$ for $x \in D$ (independent of R),

$$[4\alpha^2 e^{-2\alpha w_R(x)}]/(1-e^{-\alpha})^2$$

is bounded (independent of R and $\alpha > 0$) by a constant K_1 for all $x \in D$. Thus

$$(8) \qquad \begin{aligned} \sum W_{Rx_i}^{2}(x) &= N_2^{2}(R)\frac{\alpha^2 e^{-2\alpha w_R(x)}}{(1-e^{-\alpha})^2}\sum w_{Rx_i}^{2} \\ &\leqq K_1 N_2^{2}(R)/R^{2} \qquad \text{for all } x \in G_R^{4}. \end{aligned}$$

It follows from hypothesis (iii) that $\prod_1^n \lambda_i = 1$. But by the theorem on the geometrical and arithmetical mean,

$$(\sum \lambda_i)/n \geqq \left(\prod_1^n \lambda_i\right)^{1/n} = 1.$$

Therefore $\alpha_0 \geqq (\sum a_{ii})^n = (\sum \lambda_i)^n \geqq n^n$. Thus for any fixed $m > n^{1/2}$ we have $m\alpha_0/(n-1)^{n-1} > n\sqrt{n} \geqq 2\sqrt{2}$ and we see from the expression for η that $0 < \eta < 1$. Hence $\limsup_{R\to\infty} M(R)/R^\eta \leqq 0$ implies $\limsup_{R\to\infty} M(2R)/R \leqq 0$ and since $D_{2R} \supset G_R^2$ we can choose $R^* > R'$ sufficiently large so that for $R \geqq R^*$, $N_2^2(R)/R^2 \leqq 1/K_1$. Thus $\sum W_{Rx_i}^2(x) \leqq 1$ for all $x \in G_R^4$, $R \geqq R^*$. Consequently, by hypotheses (iv) and (vi), respectively,

$$[\sum a_{ii}(x, W_{Rx})]^n \leqq \alpha_0,$$

and

$$|f(x, W_R, W_{Rx})| \leqq \beta(\sum W_{Rx_i}^2)^\gamma \qquad \text{for all } x \in G_R^4. \tag{9}$$

Expression G of (5) is positive and by (7), (8), and (9) H is bounded by

$$\frac{2n^{1/2}\alpha_0}{(n-1)^{n-1}} - \alpha + \frac{(\sum a_{ii})^{n-1} \beta\alpha^{2\gamma-1} e^{-\alpha(2\gamma-1)w_R(x)} (\sum w_{Rx_i}^2)^{\gamma-1} N_2^{2\gamma-1}(R)}{(n-1)^{n-1}(1-e^{-\alpha})^{2\gamma-1}} \tag{10}$$

for all $x \in G_R^4$.

Since $\gamma \geqq (2-\eta)/(2-2\eta) > 1$ and $w_R(x) \geqq 1/4$ in G_R^4 we have that

$$\frac{\alpha^{2\gamma-1} e^{-\alpha(2\gamma-1)w_R(x)}}{(1-e^{-\alpha})^{2\gamma-1}} \leqq J,$$

where J is a constant independent of R and $\alpha > 0$. Therefore it follows from (6) and hypothesis (iv) that (10) is

$$\leqq \frac{2n^{1/2}\alpha_0}{(n-1)^{n-1}} - \alpha + \frac{4^{\gamma-1}\alpha_0^{(n-1)/n} \beta J N_2^{2\gamma-1}(R)}{(n-1)^{n-1}R^{2\gamma-2}}.$$

Because $\limsup_{R\to\infty} M(R)/R^\eta \leqq 0$ we may choose $R^{**} > R^*$ such that

$$\frac{N_2^{2\gamma-1}(R)}{R^{2\gamma-2}} \leqq \left[\frac{N_2(R)}{R^\eta}\right]^{2\gamma-1} < \frac{2\alpha_0[m - n^{1/2}]}{4^{\gamma-1}\alpha_0^{(n-1)/n}\beta J}$$

for all $R \geqq R^{**}$. Thus, retracting the preceding inequalities, we see that $\alpha = 2m\alpha_0/(n-1)^{n-1}$ suffices to make

$$L[W_R] - f(x, W_R, W_{Rx}) \leqq 0 \qquad \text{for all } x \in G_R^4,\ R \geqq R^{**}.$$

Throughout the remainder of this proof we consider an arbitrary but fixed $R > R^{**}$. Assume $u(x) \leqq W_R(x)$ for all $x \in D \cap S_R^4$. Now $u(x_0) = N_2(R)$ for some point $x_0 \in D \cap S_R^2$, x_0 interior to G_R^4, and $W_R(x) = N_2(R)$ on S_R^2. Therefore it follows from the maximum principle that $W_R(x) \equiv u(x)$ in G_R^4 However, $u(x) < W_R(x)$ at some points of $D \cap S_R^2 \subset G_R^4$. Consequently $u(x) > W_R(x)$ at some points of $D \cap S_R^4$, so

$$N_4(R) \geqq \min_{D \cap S_R^4} W_R(x) = N_2(R) \frac{1 - e^{-4\alpha}}{1 - e^{-\alpha}},$$

and hence, for any positive integer j,

$$N_4(3^j R) \geqq N_2(3^j R) \frac{1 - e^{-4\alpha}}{1 - e^{-\alpha}} \cdot$$

Now for every R, $G_R^4 \subset G_{3R}^2$, $G_R^4 \subset D_{4R}$, and $D_R \subset G_R^2$, so it follows by the maximum principle again that $N_2(3^j R) \geqq N_4(3^{j-1}R)$, $M(4 \cdot 3^j R) \geqq M_4(3^j R)$, and $N_2(R) \geqq M(R)$. Hence by iteration we have

$$M(4 \cdot 3^j R) \geqq M(R)\sigma^{j+1} \quad \text{where} \quad \sigma = \frac{1 - e^{-4\alpha}}{1 - e^{-\alpha}} \cdot$$

Denoting $4 \cdot 3^j R$ by R_j we have

$$M(R_j) \geqq M(R)\sigma^{j+1} = K' R_j^{\log_3 \sigma}, \qquad K' = \frac{M(R)\sigma}{(4R)^{\log_3 \sigma}},$$

from which we conclude that $\liminf_{R \to \infty} M(R)/R^\gamma > 0$. Thus we have a contradiction and we conclude that $u(x) \leqq 0$ in D.

THEOREM 2. *Assume that* (1) *satisfies conditions* (i), (ii), *and* (iii) *with* $f \equiv 0$ *and that* D *is contained in the* n*-dimensional "cone"* $x_n \geqq r \sin \alpha$ *for some* α, $0 < \alpha < \pi/2$. *Let* $u(x)$ *satisfy in* D *the inequality* $L[u] \geqq 0$ *and let the upper limit of* $u(x)$ *be nonpositive as* x *approaches any point on the boundary of* D. *Then if*

$$\limsup_{r \to \infty} \frac{M(r)}{r} \leqq 0,$$

it follows that $u(x) \leqq 0$ *throughout* D.

PROOF. Assume that $\limsup_{r\to\infty} M(r)/r \leqq 0$ and that $M(R_1) > 0$ for some R_1. By continuity of u, $u(x_0) = M(R_1)$ for some x_0 on $D \cap \{x: \|x\| = R_1\}$. For $0 < k_0 < M(R_1)/R_1$ we see that $z(x) \equiv k_0 x_n < M(R_1)$ on $\|x\| = R_1$, $x \in D$; hence $z(x_0) < u(x_0)$. Since $\limsup_{R\to\infty} M(R)/R \leqq 0$ we may, for $0 < \epsilon < k_0 \sin \alpha$, choose $R_\epsilon > R_1$ such that $M(R_\epsilon) < \epsilon R_\epsilon$. Therefore, for $\|x\| = R_\epsilon$, $x \in D$, we have

$$u(x) \leqq \epsilon R_\epsilon \leqq k_0 R_\epsilon \sin \alpha \leqq z(x).$$

Hence by the maximum principle $u(x) \leqq z(x)$ in $D \cap \{x: \|x\| \leqq R_\epsilon\}$, a contradiction. It follows that $u(x) \leqq 0$ in D.

THEOREM 3. *For dimension $n = 2$, the preceding theorem holds for any domain D contained in a half plane.*

PROOF. It suffices to consider D contained in the half plane $y > 0$. For any $\epsilon > 0$ define

$$u_\epsilon(x, y) = -\epsilon y + u(x, y)$$

and let

$$L_\epsilon[z] = a(x, y, p, q + \epsilon) z_{xx} + 2b(x, y, p, q + \epsilon) z_{xy} + c(x, y, p, q + \epsilon) z_{yy},$$

where $p = z_x$, $q = z_y$.

Choose $R_\epsilon > 0$ such that $u(x, y) \leqq (\epsilon/2)R$ for $x^2 + y^2 = R^2 \geqq R_\epsilon^2$. Then for $y \geqq R_\epsilon$ we will have

$$u_\epsilon(x, y) \leqq -\epsilon R_\epsilon + (\epsilon/2) R_\epsilon < 0.$$

Assume $u_\epsilon(0, y) > 0$ for some y', $0 < y' < R_\epsilon$. Then $u_\epsilon(0, y)$ attains a positive maximum M_ϵ at some point $(0, y_0)$, $0 < y_0 < R_\epsilon$. Define

$$v_\epsilon(x, y) = u_\epsilon(x, y) - M_\epsilon.$$

We may apply Theorem 2 to $L_\epsilon[z] = 0$ and v_ϵ in $D_1 \equiv D \cap \{\text{first quadrant}\}$ and $D_2 \equiv D \cap \{\text{second quadrant}\}$ individually to conclude that $u_\epsilon(x, y) \leqq M_\epsilon$ throughout D.

Next consider $D_{R_1} \equiv D \cap \{(x, y): x^2 + y^2 \leqq R_1^2\}$ for any $R_1 > R_\epsilon$. At the point $(0, y_0)$ interior to D_{R_1} we have $u_\epsilon(0, y_0) = M_\epsilon$; hence by the maximum principle $u_\epsilon \equiv M_\epsilon$ in D_{R_1}, which is a contradiction since $u_\epsilon(0, R_\epsilon) \leqq 0 < M_\epsilon$. Therefore $u_\epsilon(0, y) \leqq 0$ for all $y \geqq 0$, $(0, y) \in D$. Now apply Theorem 2 to $L_\epsilon[z] = 0$ and u_ϵ in D_1 and D_2 individually to conclude that $u(x, y) \leqq \epsilon y$ throughout D. Since ϵ is arbitrary we conclude that $u(x, y) \leqq 0$ throughout D.

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UNIVERSITY OF NEBRASKA