HOMOTOPICAL NILPOTENCE OF S^3

GERALD J. PORTER

In [1] Berstein and Ganea define the nilpotence of an H-space to be the least integer n such that the n-commutator is nullhomotopic. We prove that S^3 with the usual multiplication is 4 nilpotent.

Let X be an H-space. The 2-commutator $c_2: X\times X\to X$ is defined by $c_2(x, y)=xyx^{-1}y^{-1}$ where the multiplication and inverses are given by the H-space structure of X. The n-commutator $c_n: X^n\to X$ is defined inductively by $c_n=c_2(c_{n-1}\times 1)$.

Let $T_1^n(X)$ denote the subset of X^n consisting of those n-tuples $(x_1, \cdots, x_n)$ such that $x_i=*$ (the base point) for at least one i. It is well known that $c_n | T_1^n(X)\sim *$. Thus a map $\phi_n: X^n/T_1^n(X)\to X$ may be defined such that the homotopy class of ϕ_n depends only upon the homotopy class of c_n. Let Φ_n be the homotopy class of ϕ_n. Φ_n is the Samelson product and c_n is nullhomotopic if and only if $\Phi_n=0$.

The usual multiplication for S^3 is that obtained by considering S^3 to be the set of unit quaternions. With this multiplication Q_∞, infinite quaternionic projective space, is a classifying space for S^3.

Let $T: \pi(\Sigma S^{n-1}, X)\to\pi(S^{n-1}, \Omega X)$ be defined by $(Tf(s))(t)=f(t, s)$, where $s\in S^{n-1}$, $f\in\pi(\Sigma S^{n-1}, X)$, and $t\in I$. T is an isomorphism of the homotopy groups.

Samelson [3] has shown that if $j: S^4\to Q_\infty$ is inclusion, $Tj: S^3\to\Omega Q_\infty$ is an H-homomorphism which is also a homotopy equivalence. He uses this to show that $T[[j, j], j]=(Tj)_*\Phi_3$ where the product on the left is the 3-fold iterated Whitehead product. Since T is an isomorphism, to show that S^3 is 4 nilpotent it suffices to show that the four-fold iterated Whitehead product of j is zero and the three-fold product is nonzero.

Let i_4 be the identity map on S^4. Hilton [2] has shown that $0\neq[[i_4, i_4], i_4]\in\pi_{10}(S^4)$ is the image of an element in $\pi_9(S^3)$ under the suspension homomorphism. He uses this fact to prove $[[[i_4, i_4], i_4], i_4]=0$.

LEMMA 1. $[[[j, j], j], j]=0$.

PROOF. $[[[j, j], j], j]=j_*[[[i_4, i_4], i_4], i_4]=0$.

LEMMA 2. $[[j, j], j]\neq 0$.

Received by the editors May 3, 1963.

PROOF. Consider the following diagram:

$$\begin{array}{ccc} \pi_{10}(Q_\infty) & \xrightarrow{T} & \pi_9(\Omega Q_\infty) \\ \uparrow j_* & & \uparrow (Tj)_* \\ \pi_{10}(S^4) & \xleftarrow{\Sigma} & \pi_9(S^3). \end{array}$$

Let $f\in\pi_9(S^3)$, $s\in S^3$ and $t\in I$ then

$$(Tj_*\Sigma f(s))(t) = j_*\Sigma f(t, s) = j(\Sigma f)(t, s) = j(t, f(s)) = (Tj)_*f(s)(t).$$

Thus the above diagram commutes, i.e., $(Tj)_* = Tj_*\Sigma$. Since Tj is a homotopy equivalence $(Tj)_*$ is an isomorphism. By a remark above there is a $g\in\pi_9(S^3)$ such that $\Sigma g = [[i_4, i_4], i_4]$. We thus have $0\neq(Tj)_*g = Tj_*\Sigma g = T[[j, j], j]$. Therefore $[[j, j], j]\neq 0$.

THEOREM. *S^3 with the usual multiplication is* 4 *homotopy nilpotent.*

COROLLARY (TO THE PROOF). $\Sigma\Phi_3 = [[i_4, i_4], i_4]$.

REFERENCES

1. I. Berstein and T. Ganea, *Homotopical nilpotence*, Illinois J. Math. **5** (1961), 99–130.

2. P. J. Hilton, *A certain triple Whitehead product*, Proc. Cambridge Philos. Soc. **50** (1954), 189–197.

3. H. Samelson, *Groups and spaces of loops*, Comment. Math. Helv. **28** (1954), 278–287.

CORNELL UNIVERSITY