ERGODIC AND MIXING PROPERTIES OF CHEBYSHEV POLYNOMIALS

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A sequence $\{T_n: n=1, 2, \cdots\}$ of, not necessarily invertible, measurable transformations on a finite measure space, $(X, \mathfrak{B}, \mu)$, preserving the measure μ in the sense that $\mu(T_n^{-1}B)=\mu(B)$ for $B\in\mathfrak{B}$, is called *strongly mixing* if

$$\lim_{n\to\infty} \mu(T_n^{-1}A \cap B) = \frac{\mu(A)\mu(B)}{\mu(X)} \tag{1}$$

for A, $B\in\mathfrak{B}$.

Let X denote the interval $[-1, 1]$, $\mathfrak{B}$ the family of Borel subsets of X, λ Lebesgue measure and μ the equivalent measure defined by

$$\mu(B) = \frac{2}{\pi}\int_B \frac{\lambda(dx)}{\sqrt{(1-x^2)}}, \qquad B \in \mathfrak{B}.$$

The Chebyshev polynomial of degree n is defined as $T_n(x)=\cos n\theta$, where $x=\cos\theta$, $0\leqq\theta\leqq\pi$. If we put $T_0(x)=1/\sqrt{2}$ then it is well known that the sequence $\{T_k(x): k=0, 1, 2, \cdots\}$ is a complete orthonormal set (c.o.n.s.) in $L^2(X, \mathfrak{B}, \mu)$. Let T_n be the transformation defined by $T_n(x)$, then T_n is a measurable almost everywhere n to one mapping of X onto itself satisfying $T_n(T_m)=T_{nm}$ for $n, m=1, 2, \cdots$, i.e. $\{T_n: n=1, 2, \cdots\}$ is a semi-group under composition. (The semi-group property is a simple consequence of the definition of the Chebyshev polynomials.)

LEMMA. *T_n preserves the measure μ for $n=1, 2, \cdots$.*

PROOF. Consider the measure space $(X', \mathfrak{B}', \lambda')$ where X' is the interval $[0, \pi]$, $\mathfrak{B}'$ the Borel field of subsets of X' and λ' Lebesgue measure on $\mathfrak{B}'$. Let R denote the one to one measurable mapping of X onto X' determined by $x\to x'=\arccos x$, and define S_n by $S_n=RT_nR^{-1}$. If $k\pi/n\leqq x'\leqq(k+1)\pi/n$, $k=0, 1, \cdots, n-1$, we observe that $S_n(x')=nx'-k\pi$, k even; $S_n(x')=-nx'+(k+1)\pi$, k odd, and thus S_n preserves Lebesgue measure for $n=1, 2, \cdots$. A standard change of variables now reveals that

$$\int_a^b \frac{dx}{\sqrt{(1-x^2)}} = \int_{R(a)}^{R(b)} dx', \qquad -1 \leqq a < b \leqq 1,$$

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which implies that $\mu(A)=\lambda'(RA)$ for $A\in\mathfrak{B}$. Therefore $\mu(T_n^{-1}A) = \lambda'(RT_n^{-1}A) = \lambda'(RT_n^{-1}R^{-1}RA) = \lambda'(S_n^{-1}RA) = \lambda'(RA) = \mu(A)$ for $A\in\mathfrak{B}$ and $n=1, 2, \cdots$.

THEOREM. *The sequence of transformations* $\{T_n: n=1, 2, \cdots\}$ *is strongly mixing with respect to the measure* μ.

PROOF. To establish (1) it suffices to prove

$$\lim_{n\to\infty}\int_X f(T_nx)g(x)d\mu(x) = \frac{1}{\mu(X)}\int_X fd\mu(x)\cdot\int_X gd\mu(x) \tag{2}$$

for $f, g\in L^2(X, \mathfrak{B}, \mu)$, since if f is the characteristic function of A and g is the characteristic function of B, (2) becomes (1). It is sufficient to establish (2) for f, g restricted to $\{T_k: k=0, 1, 2, \cdots\}$. Having done this the fact that $\{T_k: k=0, 1, 2, \cdots\}$ is a c.o.n.s. allows us to prove (2) by a standard approximation argument. In view of the orthogonality and semi-group properties of the Chebyshev polynomials (2) is easy to verify for f and g restricted to $\{T_k: k=0, 1, 2, \cdots\}$.

An application of the theorem. Let $A_n(y_2)$ be the area under the graph of $y=T_n(x)$ contained between the lines $x=-1$, $x=1$, $y=-1$ and $y=y_2$ where $-1<y_2\leqq 1$. We wish to establish the existence and determine the value of

$$A(y_2) = \lim_{n\to\infty} A_n(y_2).$$

We proceed as follows: Let $g(x)=\sqrt{(1-x^2)}$ and $f(x)=y_2$, $y_2\leqq x$; $f(x)=x$, $-1\leqq x\leqq y_2$. Then

$$\begin{aligned} A(y_2) &= 2+\lim_{n\to\infty}\int_{-1}^{1} f(T_n(x))dx \\ &= 2+\lim_{n\to\infty}\frac{\pi}{2}\int_X f(T_n(x))g(x)d\mu = 2+\frac{\pi}{2\mu(X)}\int_X fd\mu\cdot\int_X gd\mu \\ &= 2+\frac{2}{\pi}\int_{-1}^{1} f(x)(1-x^2)^{-1/2}dx \\ &= 2+\frac{2}{\pi}\left[\int_{-1}^{y_2} x(1-x^2)^{-1/2}dx+\int_{y_2}^{1} y_2(1-x^2)^{-1/2}dx\right]. \end{aligned}$$

Thus, performing these integrations we obtain

$$A(y_2) = 2+y_2-\frac{2}{\pi}\left[(1-y_2^2)^{1/2}+y_2\arcsin y_2\right].$$

Furthermore, the limit as $n \to \infty$ of the area under $y = T_n(x)$ and between the lines $x = x_1$, $x = x_2$, $y = -1$ and $y = y_2$, where $-1 \leqq x_1 < x_2 \leqq 1$, $-1 < y_2 \leqq 1$ is $(x_2 - x_1)A(y_2)/2$. This is readily obtained by taking for $g(x)$ the product of the characteristic function of $[x_1, x_2]$ and $(1 - x^2)^{1/2}$. Therefore, the limit as $n \to \infty$ of the area under $y = T_n(x)$ and in the region bounded by the vertical lines $x = x_1$, $x = x_2$ and the continuous curves $y = y_1(x)$, $y = y_2(x)$, where $-1 \leqq x_1 < x_2 \leqq 1$, $-1 \leqq y_1(x) < y_2(x) \leqq 1$ for x in $[x_1, x_2]$ is

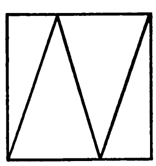
$$\frac{1}{2} \int_{x_1}^{x_2} [A(y_2(x)) - A(y_1(x))] dx.$$

REMARKS. It follows from the Theorem (in view of the semi-group property) that for $n > 1$ any particular T_n is strongly mixing in the sense that

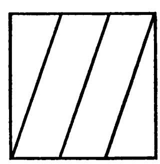
$$\lim_{k \to \infty} \mu(T_n^{-k} A \cap B) = \frac{\mu(A)\mu(B)}{\mu(X)} \cdot$$

Moreover, this implies that the T_n, $n > 1$, are *ergodic transformations*, i.e., if $T_n^{-1}A = A$, $A \in \mathfrak{B}$, then either $\mu(A) = 0$ or $\mu(A) = \mu(X)$.

It is perhaps of interest to compare the transformations S_n to the more familiar transformations $Q_n: x \to nx$ (mod 1), defined on $[0, 1]$, which also form a strongly mixing sequence. The accompanying diagrams illustrate the case $n = 3$.



S_3



Q_3

The strong mixing of $\{T_n: n = 1, 2, \cdots\}$ also can be deduced from the strong mixing of $\{Q_n: n = 1, 2, \cdots\}$ without requiring the two semi-groups of measure preserving transformations to be spatially isomorphic in the sense that there exists an isomorphism R between $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}', \lambda')$ such that $RT_nR^{-1} = Q_n$.

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