ON THE STRUCTURE OF π-REGULAR SEMIGROUPS

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1. **Introduction.** In this paper we shall investigate the structure of semigroups satisfying certain chain conditions on ideals. We say that an element α of a semigroup S is left π-regular if the chain of ideals $S \supseteq S\alpha \supseteq S\alpha^2 \supseteq S\alpha^3 \supseteq \cdots$ is finite. Equivalently, α is left π-regular if there exists an integer $n > 0$ and an element $x \in S$ such that $x\alpha^{n+1} = \alpha^n$. Right π-regularity is defined in a similar manner. The element α is π-regular if it is both left and right π-regular. Equivalently, α is π-regular if there exist an integer $n > 0$ and elements x and y in S such that $x\alpha^{n+1} = \alpha^n = \alpha^{n+1}y$. The property of π-regularity as used in this paper and in [4] corresponds to the property of strong π-regularity in the papers of Azumaya [1] and Drazin [2].

A subset G of S is said to be a group in S if it is a group under the multiplication in S. Every group in S contains a unique idempotent e of S. An element α of S is a group element of S if it belongs to some group in S. The property of being a group element is equivalent to the property of strong regularity in [1] and [2]. For each idempotent e of S the set of all elements of S belonging to some group with identity e is itself a group in S, denoted by G_e, the unique maximal group in S containing e (Kimura [3]). In [4] it is shown that an element α of S is π-regular if and only if α^n is a group element for some n. It is this property which we shall exploit in this investigation.

In §2 we give some results needed in the later sections and prove a result on the decomposition of a π-regular element in a ring. In §3 we show that a semigroup in which every element is π-regular is the union of simpler systems which we shall call proto-groups. Some conditions are given under which this union is disjoint. In the final section we examine the structure of proto-groups in terms of semigroup extensions.

2. **Some properties of π-regular semigroups and rings.** Let α be a π-regular element of the semigroup S and let $x\alpha^{n+1} = \alpha^n = \alpha^{n+1}y$. In [2], Drazin shows the existence of a unique element $\beta \in S$ having the properties: (i) $\beta\alpha = \alpha\beta$, (ii) $\beta\alpha^{n+1} = \alpha^n = \alpha^{n+1}\beta$, and (iii) $\alpha\beta^2 = \beta$. If we set $e = \alpha\beta$ then $e^2 = \alpha^2\beta^2 = \alpha \cdot \alpha\beta^2 = \alpha\beta = e$ and so e is idempotent. Moreover, we have $\beta e = \beta\alpha\beta = \beta$ and thus $\beta^n e = \beta^n$; also $\alpha^n e = \alpha^{n+1}\beta$

Received by the editors November 8, 1962 and, in revised form, May 20, 1963.

[1] This research was supported in part by the National Science Foundation under contract G-19052.

$=\alpha^n$ and $\alpha^n\beta^n=(\alpha\beta)^n=e^n=e$. Consequently, α^n and β^n are inverse elements in the group G_e. The relations $(\alpha e)\beta=(\alpha\beta)e=e^2=e$, $\alpha e\cdot e=\alpha e$ and $\beta e=\beta$ show that αe and β are inverse elements in G_e. Thus in the terms used in this paper Drazin's result may be stated:

THEOREM 2.1. *Let α be a π-regular element of a semigroup S. Then there exists an integer $n>0$, a unique idempotent e in S and a unique element β of S such that* (i) *α^n and β^n are inverse elements of G_e and* (ii) *αe and β are inverse elements of G_e. Conversely, if $\alpha\in S$ and α^n is a group element for some integer $n>0$ then α is π-regular.*

The condition that α is π-regular if and only if α^n is a group element for some n is clarified somewhat by the next theorem. Before stating this result we remark that a semigroup S can always be imbedded in the multiplicative structure of at least one ring, namely the semigroup ring of S with integral coefficients, say. Moreover if α is π-regular in S then it is π-regular in any ring containing S.

THEOREM 2.2. *Let a semigroup S be imbedded in the multiplicative structure of a ring R. Then $\alpha\in S$ is π-regular in S if and only if $\alpha=\alpha_1+\alpha_0$ where*

(i) *α_1 is a group element in S,*
(ii) *α_0 is a nilpotent element of R,*
(iii) $\alpha_0\alpha_1=\alpha_1\alpha_0=0$.

PROOF. If conditions (i)–(iii) hold for the element $\alpha\in S$ then $\alpha^n=\alpha_1^n+\alpha_0^n=\alpha_1^n$ for large enough n. Since α_1 is a group element in S so is α_1^n. Hence α^n is a group element and so α is π-regular.

Suppose now that $\alpha\in S$ is π-regular and $\alpha^n\in G_e$. Then $\alpha e\in G_e$. Set $\alpha_1=\alpha e$ and $\alpha_0=\alpha-\alpha e$. Then $\alpha=\alpha_1+\alpha_0$. Moreover $\alpha e(\alpha-\alpha e)=\alpha^2 e-\alpha^2 e=0$ and so $\alpha_1\alpha_0=\alpha_0\alpha_1=0$. Thus $\alpha^n=\alpha_1^n+\alpha_0^n$. But $\alpha^n=\alpha_1^n$ and hence $\alpha_0^n=0$, that is, α_0 is nilpotent.

We say that a semigroup or ring is π-regular if every element is π-regular. The following corollary for π-regular rings shows a resemblance to Fitting's lemma for rings of endomorphisms.

COROLLARY 2.3. *A ring R is π-regular if and only if every element is the orthogonal sum of a group element and a nilpotent element.*

3. **Unions of proto-groups.** Let S be a semigroup. For each idempotent e define T_e to be the set of all $\alpha\in S$ such that $\alpha^n\in G_e$ for some $n>0$. If S is π-regular then $S=\bigcup T_e$, the union taken over all idempotents of S, is a partition of S into disjoint subsets. If $\alpha\in T_e$ let $\langle G_e, \alpha\rangle$ be the subsemigroup of S generated by G_e and α. Then $\langle G_e, \alpha\rangle$

has the properties that it has a unique maximal group and for any $\beta \in \langle G_e, \alpha \rangle$, some power of β belongs to this group. So, in general, we say that a semigroup P is a *proto-group*

(i) if P contains a unique maximal group G;

(ii) if $\alpha \in P$ then $\alpha^n \in G$ for some $n > 0$.

We remark that a nil semigroup is a proto-group with idempotent 0. It follows immediately that every proto-group is π-regular. From the above remarks we see that a π-regular semigroup is the (not necessarily disjoint) union of proto-groups. In fact,

THEOREM 3.1. *A semigroup S is π-regular if and only if S is the union of proto-groups.*

The union will be disjoint if and only if each of the T_e defined above is a subsemigroup of S. In this case T_e will itself be a proto-group with unique maximal group G_e. We have not yet discovered a satisfactory set of necessary and sufficient conditions that a semigroup be a disjoint union of proto-groups. However, two sufficient conditions are given in

THEOREM 3.2. *If S is commutative or if all the idempotents of S are primitive then S is π-regular if and only if S is a disjoint union of proto-groups.*

PROOF. Suppose S is commutative. If $\alpha, \beta \in T_e$ then $\alpha^m, \beta^n \in G_e$ for some $m, n > 0$. Set $N = \max(m, n)$. Then $(\alpha\beta)^N = \alpha^N \beta^N \in G_e$ and so $\alpha\beta \in T_e$. Thus, T_e is a proto-group.

Now suppose that every idempotent of S is primitive; that is, if e and f are idempotents of S such that $ef = fe = e$ then $e = f$. Let $\alpha, \beta \in T_e$ and $\alpha\beta \in T_f$, e and f idempotents of S. By Theorem 2.1, $\alpha e = e\alpha$, $\beta e = e\beta$ and $(\alpha\beta)^n \in G_f$ for some $n > 0$. Hence, $(\alpha\beta)^n e = [(\alpha e)(\beta e)]^n = e(\alpha\beta)^n \in G_e$. Let γ be the inverse of $(\alpha\beta)^n e$ in G_e; then, since $(\alpha\beta)^n \in G_f$, $e = \gamma e(\alpha\beta)^n = \gamma e(\alpha\beta)^n f = \gamma(\alpha\beta)^n ef = ef$. Similarly, $e = fe$. Since all idempotents are primitive this implies $e = f$ and so $\alpha\beta \in T_e$. Therefore, T_e is a proto-group.

If e and f are idempotents of S, then we say that f *dominates* e, or $f \succ e$, if $ef = fe = e$. Thus, an idempotent e is primitive if it is minimal in the sense that e dominates no idempotent other than itself. We shall say that e is a maximal idempotent if no idempotent other than e dominates e.

The following corollary can be extracted from the above proof.

COROLLARY 3.3. *If α and $\beta \in T_e$ then $\alpha\beta \in T_f$ where $f \succ e$. In particular, if e is a maximal idempotent of S then T_e is a proto-group.*

4. The structure of proto-groups.

THEOREM 4.1. *Let P be a proto-group with unique maximal group G. Then G is the unique minimal ideal of P. Moreover, the Rees factor semigroup P/G is a nil semigroup.*

PROOF. Let I be any ideal of P and $\alpha\in I$. If e is the identity of G then $\alpha e\in I$ and $\alpha e\in G$. Hence, I contains an element of G and therefore I contains all of G. If $\alpha\in P$ and $\beta\in G$, then $\alpha\beta=\alpha(e\beta)=(\alpha e)\beta\in G$ and so G is an ideal of P contained in every ideal of P.

Since G is an ideal we may form the Rees factor semigroup (Rees [5]) P/G. For any $\alpha\in P$, $\alpha^n\in G$ for some n. Hence, we have $\bar{\alpha}^n=\bar{0}$ where $\bar{\alpha}$ is the image of α in P/G. Thus, P/G is nil.

Let S be a semigroup, P a proto-group with idempotent e and $\sigma\colon S\to P$ a homomorphism of S onto P. We define the kernel N of σ to be the set of all $x\in S$ such that $\sigma(x)=e$. Clearly, N is a subsemigroup of S.

Let N be a semigroup and P a proto-group. A semigroup S is said to be an extension of N by P if $N\subseteq S$ and if there exists a homomorphism $\sigma\colon S\to P$ with kernel N. We shall say that S is a split extension of N by P if there exists a subset P' of S such that $\sigma\colon P'\to P$ is an isomorphism of P' onto P. It follows that $N\cap P'=\{e'\}$ where e' is the unique inverse image of e in P'. Thus, we may (and will) assume that in a split extension $P\subseteq S$.

THEOREM 4.2. *Every proto-group is a split extension of a nil semigroup by a group.*

PROOF. Let G be the unique maximal group in the proto-group P and let e be its idempotent. Then $\alpha\to\alpha e$ is a homomorphism of P onto G leaving the elements of G fixed. Thus, P is a split extension of the kernel of this homomorphism by G. The kernel N consists of all $\alpha\in P$ such that $\alpha e=e$. If $\alpha\in N$ then there exists an integer $n>0$ such that $\alpha^n\in G_e$. Hence, $\alpha^n=\alpha^n e=(\alpha e)^n=e$. Thus we see that e is the zero element of N and that N is nil.

It is not true, however, that every split extension of a nil semigroup by a group is a proto-group. But we can prove

THEOREM 4.3. *If P is a split extension of a nil semigroup N by a group G, then G is the unique maximal group and unique minimal ideal of P. Moreover, if $\sigma\colon P\to G$ is the homomorphism giving the extension and e is the identity of G, then $\sigma(\alpha)=\alpha e$ for all $\alpha\in P$.*

PROOF. We assume that G is imbedded in P and $N\cap G=\{e\}$. Since $e\in N$ and $e^n=e$ for all n it follows that e must be the zero element of N. Let $\alpha \in P$. Since $\sigma(\alpha) \in G$ there exists $\beta \in G$ such that $\sigma(\alpha)\beta = \beta\sigma(\alpha) = e$. But $\sigma(\beta) = \beta$ and so $\sigma(\alpha\beta) = \sigma(\beta\alpha) = e$; thus, $\alpha\beta$ and $\beta\alpha \in N$. Hence we have $\alpha\beta = \alpha\beta e = e = e\beta\alpha = \beta\alpha$. Since β is the inverse of $\sigma(\alpha)$ we have $\alpha e = \alpha(\beta\sigma(\alpha)) = (\alpha\beta)\sigma(\alpha) = e\sigma(\alpha) = \sigma(\alpha)$. Thus, $\sigma(\alpha) = \alpha e = e\alpha$ for all $\alpha \in P$. It follows that G is the unique maximal group and unique minimal ideal of P.

COROLLARY 4.4. *A split extension of a nil semigroup by a periodic group is a proto-group.*

PROOF. If $\alpha \in P$, the split extension of the nil semigroup N by the periodic group G, then $\alpha e \in G$. Hence, $(\alpha e)^n = \alpha^n e = e$ for some positive n. Therefore, $\alpha^n \in N$. But N is nil and so $(\alpha^n)^m = e \in G$. Therefore, P is a proto-group.

By using the property of a proto-group given in 4.1 and the result of 4.3, we can give necessary and sufficient conditions that a semigroup be a proto-group.

THEOREM 4.5. *Let P be a split extension of a nil semigroup N by a group G. Then P is a proto-group if and only if the Rees factor semigroup P/G is a nil semigroup.*

PROOF. The necessity has already been shown. If P is a split extension of G by N then G is an ideal of P and so P/G is defined. If $\alpha \in P$ then $\bar{\alpha}^n = 0$ in P/G for some n where $\bar{\alpha}$ denotes the image of α in P/G. This is equivalent to saying $\alpha^n \in G$. Hence, P is a proto-group.

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