NOTE ON POSITIVE LINEAR OPERATORS

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Let $\mathfrak{L}$ be a topological linear space, and let $\mathfrak{C}$ be a closed convex cone in $\mathfrak{L}$ such that $\mathfrak{L}=\mathfrak{C}-\mathfrak{C}$, i.e., such that $\mathfrak{L}=(\mathfrak{L}, \mathfrak{C})$ is a *directed* topological linear space.[1]

LEMMA 1. *If $\gamma_n\to\gamma$, $f_n\to f$, $g_n\to g$, and $\gamma_n f_n \geqq g_n$ for $n=1, 2, 3, \cdots$, then $\gamma f\geqq g$.*

For, $\gamma f-g=\lim(\gamma_n f_n-g_n)\in\mathfrak{C}$ since $\mathfrak{C}$ is closed.

COROLLARY. *$\mathfrak{L}=(\mathfrak{L}, \mathfrak{C})$ is an Archimedean directed vector space.*

For, if $0\leqq g\leqq\gamma_n f$ where $\gamma_n\downarrow 0$, then $g\leqq 0f=0$.

Conversely, any Archimedean directed vector space $\mathfrak{L}=(\mathfrak{L}, \mathfrak{C})$, given its intrinsic order or relative uniform topology, is a topological linear space in which $\mathfrak{C}$ is a closed convex cone with $\mathfrak{L}=\mathfrak{C}-\mathfrak{C}$. Hence our results apply to vector lattices in their usual intrinsic topologies.

Now let $\theta(f, g)$ be the projective quasi-metric on $\mathfrak{C}-0$ defined by

$$\theta(f, g) = \ln(\alpha_0\beta_0), \text{ where} \tag{1}$$

$$\alpha_0 = \inf\{\alpha \mid \alpha f \geqq g\}, \qquad \beta_0 = \inf\{\beta \mid \beta g \geqq f\}. \tag{1'}$$

LEMMA 2. *The projective quasi-metric $\theta(f, g)$ is a lower-semicontinuous function on $\mathfrak{C}\times\mathfrak{C}$.*

PROOF. Let $f_n\to f$ and $g_n\to g$ in $\mathfrak{C}$, and let α_n and β_n be the *least* numbers such that $\alpha_n f_n\geqq g_n$ and $\beta_n g_n\geqq f_n$. These exist by Lemma 1 and are positive since $\mathfrak{L}$ is Archimedean, and $\theta(f_n, g_n)=\theta_n$ satisfies $e^{\theta_n}=\alpha_n\beta_n$. Let $\boldsymbol{\theta}=\liminf\theta_n$. The case $\boldsymbol{\theta}=\infty$ is trivial, since it imposes no restriction on $\theta(f, g)$. Moreover, by restricting attention to a subsequence, we can reduce to the case $\boldsymbol{\theta}=\lim\theta_n$.

This may increase the values of $\boldsymbol{\alpha}=\liminf\alpha_n$ and $\boldsymbol{\beta}=\liminf\beta_n$. But both $\boldsymbol{\alpha}>0$ and $\boldsymbol{\beta}>0$ since, by Lemma 1, $\boldsymbol{\alpha} f\geqq g>0$ and $\boldsymbol{\beta} g\geqq f$ where $\boldsymbol{\alpha}\boldsymbol{\beta}\leqq e^{\boldsymbol{\theta}}$ and $\mathfrak{L}$ is Archimedean. It follows that $\boldsymbol{\alpha}<+\infty$ and $\boldsymbol{\beta}<+\infty$. Now extract a subsequence $\alpha_n\to\boldsymbol{\alpha}$; it will follow that $\beta_n=e^{\theta_n}/\alpha_n\to e^{\boldsymbol{\theta}}/\boldsymbol{\alpha}\geqq\boldsymbol{\beta}$. Moreover, by Lemma 1, $\boldsymbol{\alpha} f\geqq g$ and $(e^{\boldsymbol{\theta}}/\boldsymbol{\alpha})g\geqq f$, whence

$$\theta(f, g) \leqq \ln[\boldsymbol{\alpha}(e^{\boldsymbol{\theta}}/\boldsymbol{\alpha})] = \boldsymbol{\theta} = \liminf\theta(f_n, g_n). \tag{2}$$

This proves the lemma.

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[1] For the terminology used here, and the background of the present discussion, see [1].

COROLLARY 1. *For any continuous strictly positive linear operator* P *on* $(\mathfrak{L}, \mathfrak{C})$, *the function* $\delta(f)=\theta(f, fP)$ *is lower-semicontinuous on* $\mathfrak{C}$.

Note that, in the (intrinsic) relative uniform topology on any Archimedean directed vector space, any positive linear operator is continuous [1, Lemma 2].

THEOREM 1. *Let* P *be any completely continuous strictly positive linear operator on the partially ordered*[2] *Banach space* $\mathfrak{L}=(\mathfrak{L}, \mathfrak{C})$, *and let* $\delta_0=\inf_{\mathfrak{C}} \delta(f)$. *Then the set of nonzero* $f\in\mathfrak{C}$ *where* $\delta(f)=\delta_0$ *is a nonvoid closed cone* Δ_0, *invariant under* P.

PROOF. Let S be the unit sphere. Since $\delta(\lambda f)=\delta(f)$ for any positive scalar λ, $\delta_0=\inf_{S\cap\mathfrak{C}} \delta(f)$. Moreover, since $\theta(fP, fP^2)\leqq\theta(f, fP)$ by [1, (11)], $\delta(fP)\leqq\delta(f)$ and so $\delta_0=\inf_{SP\cap\mathfrak{C}} \delta(f)$. But by hypothesis, $SP\cap\mathfrak{C}$ has a compact closure $\mathfrak{K}$. Hence every sequence of $f_n\in S\cap\mathfrak{C}$ with $\delta(f_n)=\theta(f_n, f_nP)\downarrow\delta_0$ has a limit point $g\in\mathfrak{K}$, where $g\in\mathfrak{C}$ since $\mathfrak{C}$ is closed, and so some $\alpha g\in S\cap\mathfrak{C}$, $\alpha>0$. Further, by Lemma 2:

$$\theta(\alpha g, \alpha gP) \leqq \theta(g, gP) \leqq \liminf \theta(f_nP, f_nP^2) \leqq \liminf \theta(f_n, f_nP) = \delta_0,$$

since $\theta(hP, kP)\leqq\theta(h, k)$ for any $h, k\in\mathfrak{C}$. Consequently $\delta(f)$ *assumes* the values δ_0 on a nonvoid closed set Δ_0. This set is a cone invariant under P since, as noted above, $\delta(fP)\leqq\delta(f)$ and $\delta(\lambda f)=\delta(f)$.

Obviously, the Theorem of Jentzsch refers to the special case $\delta_0=0$. In this case, Δ_0 is the cone of invariant directions, and the preceding argument is closely related to proofs of Jentzsch's Theorem by Kreĭn and Rutman.

It would be interesting to extend Theorem 1 to Archimedean directed vector spaces which are not Banach spaces. It would be even more interesting to know more about the structure of the set Δ_0. In this connection, the following example is relevant.

EXAMPLE 1. Let $\mathfrak{L}$ be the vector lattice of all continuous functions on [0, 1], let $\mathfrak{C}$ consist of all nonnegative $f(x)$, and let $P[f(x)] = p(x)f(x)$, where $p(x)$ is a positive continuous function.

One easily verifies that, in Example 1, P is an *isometry* for the projective quasi-metric:

$$\theta(f, g) = \theta(fP, gP) \qquad \text{for all } f, g \in \mathfrak{C}. \tag{3}$$

Moreover

$$\theta(f, fP) = m = \ln[\sup p(x)]/[\inf p(x)] \tag{4}$$

[2] We continue to assume that $\mathfrak{C}$ is closed and that $\mathfrak{L}=\mathfrak{C}-\mathfrak{C}$.

is independent of f: all elements of $\mathfrak{C}$ are moved through the same distance. The only transformations of Euclidean space which satisfy (3)–(4) are *translations*, but the analogy is very poor: the projective quasi-metric defined by (1)–(1′) has little in common with Euclidean distance.

Unfortunately, Example 1 (in which $\Delta_0 = \mathfrak{C}$) does not seem to be typical. Not only is $\mathfrak{C}P$ not compact, but in general the set Δ_0 is not even convex, on the connected components of $\mathfrak{C}$. This is shown by the following example constructed by Mr. Alan G. Waterman.[3]

EXAMPLE 2 (WATERMAN). Let P be the linear operator

$$(5) \quad (x, y, u, v) \to (x, (x+v)/(k+1), v/(k-1) + (k-2)u/(k-1), v).$$

Let $f = (1, 1, 1, k)$ and $g = (k, 2, 1, 1)$. Then, for any $k > 2$, we have $\delta(f) = \delta(g) = \ln 2$, but

$$(6) \qquad \delta(f+g) = \ln(9/4) > \max\{\delta(f), \delta(g)\}.$$

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REFERENCES

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