the power series for $(\bar{a}-bz)^{-1}$, however, converges as previously required, hence we have that $U(\sigma)zU(\sigma^{-1})=(az-\bar{b})/(\bar{a}-bz)=\sigma^{-1}z$.

References

1. G. W. Mackey, *Imprimitivity for representations of a locally compact group*. I, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 537–545.

2. ———, *Unitary representations of group extensions*. I, Acta Math. **99** (1958), 265–311.

3. R. J. Blattner, *On induced representations*. II. *Infinitesimal induction*, Amer. J. Math. **83** (1961), 499–512.

4. V. Bargmann, *Irreducible unitary representations of the Lorenz group*, Ann. of Math. (2) **48** (1947), 568–640.

Illinois Institute of Technology

ON WANDERING SUBSPACES FOR UNITARY OPERATORS

J. B. ROBERTSON[1]

Let V be a unitary operator on a complex Hilbert space H. X is said to be a wandering subspace for V if it is a subspace of H such that $V^m(X)\perp V^n(X)$ for all $m\neq n$. The purpose of this note is to study the relation between two wandering subspaces X and Y satisfying $\sum_{k=-\infty}^{\infty} V^k(X)\subseteq\sum_{k=-\infty}^{\infty} V^k(Y)$.

Theorem 1. *Let X and Y be wandering subspaces for a unitary operator V such that*:

(a) $\sum_{k=-\infty}^{\infty} V^k(X)\subseteq\sum_{k=-\infty}^{\infty} V^k(Y)$,

(b) $\dim(X)=\dim(Y)<\infty$.

Then $\sum_{k=-\infty}^{\infty} V^k(X)=\sum_{k=-\infty}^{\infty} V^k(Y)$.

Proof. Let $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ be orthonormal bases for X and Y, respectively. Since $x_i\in\sum_{k=-\infty}^{\infty} V^k(Y)$ we have

$$x_i = \sum_{r=1}^{n}\sum_{k=-\infty}^{\infty} a_{ir\,k}V^k(y_r), \qquad a_{ir\,k} = (x_i, V^k(y_r)),$$

$$\sum_{r=1}^{n}\sum_{k=-\infty}^{\infty} |a_{ir\,k}|^2 < \infty, \quad i = 1, \cdots, n.$$

It follows that

Received by the editors October 30, 1963.

[1] This research was supported in part by National Science Foundation Grant No. GP 2234 while the author was at Cornell University.

$$(1)\qquad \delta_{m0}\delta_{ij} = (x_i,\ V^m(x_j)) = \sum_{r=1}^{n}\sum_{k=-\infty}^{\infty} a_{ir\ k}\bar{a}_{jr\ k-m}.$$

Since $\sum_{k=-\infty}^{\infty}|a_{ir\ k}|^2<\infty$, there exists a function $f_{ir}(\cdot)\in L(0,\ 2\pi)$ such that $f_{ir}(\theta)\sim\sum_{k=-\infty}^{\infty} a_{ir\ k}e^{ik\theta}$. (1) may now be restated as

$$(1')\qquad \sum_{r=1}^{n} f_{ir}(\theta)\bar{f}_{jr}(\theta) = \delta_{ij} \text{ a.e. (Leb.)},\quad i, j = 1, \cdots, n.$$

By the usual theory of $n\times n$ matrices, (1′) is equivalent to

$$(2')\qquad \sum_{i=1}^{n} \bar{f}_{ir}(\theta)f_{is}(\theta) = \delta_{rs} \text{ a.e. (Leb.)},\quad r, s = 1, \cdots, n.$$

(2′), in turn, may be restated in terms of the coefficients as

$$(2)\qquad \delta_{m0}\delta_{rs} = \sum_{i=1}^{n}\sum_{k=-\infty}^{\infty} \bar{a}_{ir\ k}a_{is\ k-m}.$$

Now let Q be the projection operator onto the space $\sum_{k=-\infty}^{\infty} V^k(X)$. Since $(y_r,\ V^k(x_i))=\bar{a}_{ir\ -k}$, we have

$$Q(y_r) = \sum_{i=1}^{n}\sum_{k=-\infty}^{\infty} \bar{a}_{ir\ -k}V^k(x_i),\quad r = 1, \cdots, n.$$

From (2) we obtain

$$\|Q(y_r)\|^2 = \sum_{i=1}^{n}\sum_{k=-\infty}^{\infty} |a_{ir\ -k}|^2 = 1,\quad r = 1, \cdots, n.$$

Therefore, $\|y_r\|=\|Q(y_r)\|$ which implies $y_r=Q(y_r)\in\sum_{k=-\infty}^{\infty} V^k(X)$. Thus $\sum_{k=-\infty}^{\infty} V^k(X)=\sum_{k=-\infty}^{\infty} V^k(Y)$. Q.E.D.

Theorem 1 is clearly false if $\dim(Y)=\infty$, but the following corollary, due to Halmos [1, Lemma 4], is true for all dimensions.

COROLLARY. *Let X and Y be wandering subspaces for a unitary operator V such that $\sum_{k=-\infty}^{\infty} V^k(X)\subseteq\sum_{k=-\infty}^{\infty} V^k(Y)$. Then* $\dim(X)\leqq\dim(Y)$.

PROOF. If the dimension of Y is not finite then, since $X\subseteq\sum_{k=-\infty}^{\infty} V^k(Y)$, $\dim(X)\leqq\dim(\sum_{k=-\infty}^{\infty} V^k(Y))=\dim(Y)$ as desired. Next let $\dim(Y)=n<\infty$ and suppose $\dim(X)>n$. If $x_1, \cdots, x_{n+1}$ are orthonormal vectors in X and if $\tilde{X}$ is the space spanned by $x_1, \cdots, x_n$, then $\sum_{k=-\infty}^{\infty} V^k(\tilde{X})=\sum_{k=-\infty}^{\infty} V^k(Y)$ by Theorem 1. Thus $x_{n+1}\in(\sum_{k=-\infty}^{\infty} V^k(\tilde{X}))^{\perp}\cap(\sum_{k=-\infty}^{\infty} V^k(Y))=\{0\}$ which contradicts the assumption that x_{n+1} is normal. Thus $\dim(X)\leqq n$. Q.E.D.

It follows from the corollary, of course, that if $\sum_{k=-\infty}^{\infty} V^k(X) = \sum_{k=-\infty}^{\infty} V^k(Y)$, then $\dim(X)=\dim(Y)$. The final theorem shows that finite-dimensional wandering subspace may be increased to the maximum.

THEOREM 2. *Let X and Y be wandering subspaces for a unitary operator V such that*:

(a) $\sum_{k=-\infty}^{\infty} V^k(X) \subseteq \sum_{k=-\infty}^{\infty} V^k(Y)$,

(b) $\dim(Y) < \infty$.

Then there exists a wandering subspace $\tilde{X}$ such that:

(1) $X \subseteq \tilde{X}$,

(2) $\sum_{k=-\infty}^{\infty} V^k(\tilde{X}) = \sum_{k=-\infty}^{\infty} V^k(Y)$.

PROOF. Let $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ be orthonormal bases for X and Y, respectively, and suppose $m<n$. As before, let

$$x_i = \sum_{r=1}^{n} \sum_{k=-\infty}^{\infty} a_{ir\,k} V^k(y_r),$$

$$f_{ir}(\theta) \sim \sum_{k=-\infty}^{\infty} a_{ir\,k} e^{ik\theta}, \quad i = 1, \cdots, m,\ r = 1, \cdots, n.$$

Then (cf. (1')) $\boldsymbol{f}_i(\theta) = (f_{i1}(\theta), \cdots, f_{in}(\theta))$, $i=1, \cdots, m$, are orthonormal vectors in C^n for almost all θ. We can extend this to an orthonormal basis of C^n, $(\boldsymbol{f}_1(\theta), \cdots, \boldsymbol{f}_n(\theta))$, and this can be done measurably. (E.g., apply the Gram-Schmidt process to $(\boldsymbol{f}_1(\theta), \cdots, \boldsymbol{f}_m(\theta), \boldsymbol{e}_1, \cdots, \boldsymbol{e}_n)$, where $\boldsymbol{e}_i = (\delta_{ij})_{j=1}^{n}$.) Since $\boldsymbol{f}_i(\theta)$ is normal almost everywhere, its component functions $f_{ir}(\cdot)$ are in $L_2(0, 2\pi)$. Let, therefore,

$$f_{ir}(\theta) \sim \sum_{k=-\infty}^{\infty} a_{ir\,k} e^{ik\theta}, \quad i, r = 1, \cdots, n.$$

Then, as in the proof of Theorem 1, the space $\tilde{X}$ spanned by the orthonormal vectors

$$x_i = \sum_{r=1}^{n} \sum_{k=-\infty}^{\infty} a_{ir\,k} V^k(y_r), \quad i = 1, \cdots, n,$$

has the desired properties. Q.E.D.

The following example shows that the assumption $\dim(Y) < \infty$ is essential for Theorem 2.

EXAMPLE. Let $y_1, y_2, \cdots$ be a complete orthonormal set of vectors for a wandering subspace Y of a unitary operator V. Let $l_2 = \{a = (a_1, a_2, \cdots): \sum_{k=1}^{\infty} |a_k|^2 < \infty,\ a_k \text{ complex numbers}\}$, and let

$$f_{ir}(\theta) = \begin{cases} \delta_{ir} & \text{if } 0 \leqq \theta \leqq \pi \\ \delta_{i+1r} & \text{if } \pi < \theta \leqq 2\pi \end{cases} \sim \sum_{k=-\infty}^{\infty} a_{ir\,k} e^{ik\theta}.$$

Then $f_i(\theta) = (f_{ir}(\theta))_{r=1}^{\infty}$ for $0 \leqq \theta \leqq 2\pi$ are orthonormal vectors in l_2. Thus the space X spanned by the orthonormal vectors

$$x_i = \sum_{r=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{ir\,k} V^k(y_r), \quad i = 1, 2, \cdots,$$

is a wandering subspace contained in $\sum_{k=-\infty}^{\infty} V^k(Y)$. X cannot be extended as in Theorem 2 since this would imply the existence of a function f: $[0, 2\pi] \to l_2$ such that $(f(\theta), f(\theta)) = 1$ almost everywhere and that $(f(\theta), f_i(\theta)) = 0$, $i = 1, 2, \cdots$, almost everywhere. But $f_i(\theta)$, $i = 1, 2, \cdots$, span l_2 for θ between 0 and π. On the other hand, if

$$f_r(\theta) = \begin{cases} 0 & \text{if } 0 \leqq \theta \leqq \pi \\ \delta_{1r} & \text{if } \pi < \theta \leqq 2\pi \end{cases} \sim \sum_{k=-\infty}^{\infty} b_{rk} e^{ik\theta},$$

$$f(\theta) = (f_r(\theta))_{r=1}^{\infty},$$

then

$$x = \sum_{r=1}^{\infty} \sum_{k=-\infty}^{\infty} b_{rk} V^k(y_r) \in \sum_{k=-\infty}^{\infty} V^k(Y)$$

is clearly orthogonal to $\sum_{k=-\infty}^{\infty} V^k(X)$, since $f(\theta)$ is orthogonal to $f_r(\theta)$, $r = 1, 2, \cdots$. Thus $\sum_{k=-\infty}^{\infty} V^k(X) \neq \sum_{k=-\infty}^{\infty} V^k(Y)$.

Reference

1. P. Halmos, *Shifts on Hilbert space*, J. Reine Angew. Math. 208 (1961), 102–112.

Indiana University and
Cornell University