PERSPECTIVITY IN PROJECTION LATTICES[1]

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Let A be a w^*-algebra and L its projection lattice. It is well known [3] that projections perspective in L are linked by a partial isometry in A, and that these two notions coincide precisely when the algebra A is finite. We show in this paper that, in any w^*-algebra, perspectivity enjoys virtually every property, other than additivity, of the relation $\sim$ of partial isometry equivalence. These properties are established by means of

THEOREM 1. *Projections $p, q \in L$ are perspective in L if and only if they are unitarily equivalent in A.*

In one direction this is immediate, for if $r \in L$ is a common complement for p and q, then $r' = 1 - r$ is a common complement for p' and q', $p \sim q$, $p' \sim q'$, and p and q are unitarily equivalent.

We begin the converse with two lemmas (valid in any orthomodular lattice L) concerning the additivity of perspectivity. Recall that an orthomodular lattice is a lattice with an orthocomplementation $a \rightarrow a'$ such that $b = a + a' \cap b$ whenever $a \leqq b$. (The symbol $+$ will be used for the lattice join if the summands are orthogonal.) It is easily seen that this condition is equivalent to the following:

$$a \leqq c \quad \text{implies} \quad (a + b) \cap c = a + b \cap c.$$

LEMMA 1. *If $a, b, c, d \in L$ and $a \cup c \perp b \cup d$, then $(a+b) \cap (c+d) = a \cap c + b \cap d$.*

To prove this one can imitate the proof in [5, Theorem 1.2], observing that our assumption permits replacement of the modular law by the restricted version of it given above.

LEMMA 2. *If $a, b, c, d, e, f \in L$, a and b are perspective in $[0, e]$, c and d are perspective in $[0, f]$, and $e \perp f$, then $a + c$ and $b + d$ are perspective in $[0, e+f]$, as are $a+f$ and $b+f$.*

PROOF. Let x be a common complement for a and b in $[0, e]$, y for c and d in $[0, f]$, and put $z = x + y$. Plainly, $a \cup c \cup z = e + f = b \cup d \cup z$, and, since $a \cup x \perp c \cup y$, we have

$$(a \cup c) \cap z = (a + c) \cap (x + y) = a \cap x + c \cap y = 0$$

Received by the editors February 7, 1964.

[1] This work was supported by a grant from the National Science Foundation.

by Lemma 1. Similarly $(b\cup d)\cap z=0$, and the second part of the lemma follows in an analogous fashion.

For the remainder of the paper L will denote the projection lattice of a w^*-algebra on a Hilbert space H. We are indebted to Professors Halperin and Kaplansky for simplifications of several of the subsequent proofs.

LEMMA 3. *If p, $q\in L$, $p\cap q=0$, and $p\sim q$, then p and q are perspective in $[0, p\cup q]$.*

PROOF. Following [2; p. 9], if $u\in A$ implements the equivalence of p and q and λ is a complex number with $|\lambda|<1$, then the range projection of $(1+\lambda u)p$ has the required properties.

LEMMA 4. *If p, q, $r\in L$ satisfy $q\sim p\leqq q$ and $p\precsim r\leqq q'$, then p and q are perspective in $[0, q+r]$.*

PROOF. $p\sim s\leqq r$, and let u, v be partial isometries in A with $u^*u=q$, $uu^*=s$, $v^*v=q$, and $vv^*=p$. For $n\geqq 1$ denote by c_n the range projection of $(v^n+(1/n)(1+uv^n))(q-p)$, and let $c=\bigcup c_n$. Of course $c\in L$, and it will be shown that c is a common complement for p and q in $[0, p\cup c]$ by showing (1) $q\leqq p\cup c$ and (2) $q\cap c=0$.

Let $e\in L$, $e\perp p\cup c$. Then, for each $n\geqq 1$, $e(v^n+(1/n)(1+uv^n))(q-p)=0$, whence $e(1+uv^n)(q-p)=0$ and so $e(q-p)=-(1/n)\sum_{k=1}^{n} euv^k(q-p)$. For each $\alpha\in H$ we have

$$\|e(q-p)\alpha\| = \frac{1}{n}\left\|\sum_{k=1}^{n} euv^k(q-p)\alpha\right\| \leqq \frac{1}{n}\sqrt{n}\|(q-p)\alpha\|,$$

so that $e(q-p)=0$, $e\perp q$, and $q\leqq p\cup c$.

To prove (2) suppose that α is an element in the range of $q\cap c$. For any $\epsilon>0$ there are elements α_k in the range of $q-p$ such that

$$(*)\qquad \left\|\alpha - \sum\left(v^k + \frac{1}{k}(1+uv^k)\right)\alpha_k\right\| < \epsilon.$$

Hence, since $q'\alpha=0$, we have

$$\left\|\sum \frac{1}{k}uv^k\alpha_k\right\| = \left(\sum \frac{1}{k^2}\|\alpha_k\|^2\right)^{1/2} < \epsilon.$$

Denoting by p_k the range projection of $v^k(q-p)$, $(*)$ implies $\|p_k\alpha - v^k\alpha_k\| < \epsilon$, and, therefore,

$$\|p_k\alpha\| \leqq \epsilon + \|v^k\alpha\| \leqq \epsilon + k\epsilon.$$

Since ϵ is arbitrary, $p_k\alpha=0$. Again $(\sum p_k)\alpha=p\alpha$, for, by $(*)$,

$\|p\alpha-\sum v^k\alpha_k\|<\epsilon$ and $\|(\sum p_k)\alpha-\sum v^k\alpha_k\|<\epsilon$. Thus $p\alpha=0$, whence $\|\sum v^k\alpha_k\|=(\sum\|\alpha_k\|^2)^{1/2}<\epsilon$. Finally, $\|(q-p)\alpha-\sum(1/k)\alpha_k\|<\epsilon$ by $(*)$, so

$$\|(q-p)\alpha\| \leqq \epsilon + \left\|\sum \frac{1}{k}\alpha_k\right\| \leqq \epsilon + \left[(\sum\|\alpha_k\|^2)\left(\sum \frac{1}{k^2}\right)\right]^{1/2} \leqq K\epsilon,$$

K fixed. Therefore, $q\alpha=0$, $\alpha=0$, and $q\cap c=0$.

It is readily verified that $p\cup c\leqq q+r$, and the proof is completed by enlarging c to a common complement for p and q in $[0, q+r]$.

LEMMA 5. *If a, b, $c\in L$ satisfy $a+b+c\sim a+b$, then there exist c_1, $c_2\in L$ such that $c=c_1+c_2$, $a+c_1\sim a$, and $b+c_2\sim b$.*

PROOF. Split the algebra into two summands on which, respectively, $a\precsim b$ and $b\precsim a$. On the first summand we can assume that b is purely infinite, for b finite implies a finite, $a+b$ finite, and $c=0$. Hence $b=b_1+b_2$ with $b\sim b_1\sim b_2$, so

$$b + c \leqq a + b + c \sim a + b \precsim b_1 + b_2 = b$$

and we have $b+c\sim b$. On the summand with $b\precsim a$ we have, similarly, $a+c\sim a$.

LEMMA 6. *If a, b, $x\in L$ satisfy $a\leqq b$ and $a\precsim x\precsim b$, then there exists $y\in L$ such that $y\sim x$ and $a\leqq y\leqq b$.*

PROOF. Let $x=c+u$ with $a\sim c$, so that $c+u\precsim a+(b-a)$. Split the algebra into two summands on which, respectively, $u\precsim b-a$ and $b-a\precsim u$. On the first we have $u\sim w\leqq b-a$, so the element $y=a+w$ meets our requirements. If $b-a\precsim u$ we have $b=a+(b-a)\precsim c+u=x\precsim b$, so $b\sim x$ and $y=b$ serves.

PROOF OF THEOREM 1. Let p, $q\in L$ be unitarily equivalent. Drop first to a direct summand in which $p\cap q\precsim p'\cap q'$, and split this into two further summands in which $p-p\cap q\precsim q-p\cap q$ and $q-p\cap q\precsim p-p\cap q$, respectively. In the first of these we wish to locate $s\leqq p'\cap q$ such that $p-p\cap q\sim q-p\cap q-s$. To this end we note that the elements $a=p-p\cap q-p\cap q'$ and $b=q-p\cap q-p'\cap q$ satisfy $a\cap b'=a'\cap b=0$. Hence b' is a common complement for a and b, $a\sim b$, and

$$q - p\cap q - p'\cap q \precsim p - p\cap q \precsim q - p\cap q.$$

Applying Lemma 6 there exists $r\sim p-p\cap q$ such that

$$q - p\cap q - p'\cap q \leqq r \leqq q - p\cap q$$

and letting $r+s=q-p\cap q$ we have $s\leqq p'\cap q$ and $p-p\cap q\sim q-p\cap q-s$ as desired. Now

$$r+p\cap q\sim (p-p\cap q)+p\cap q\sim q=r+p\cap q+s$$

so, by Lemma 5, we have $s=s_1+s_2$ with $r+s_1\sim r$ and $p\cap q+s_2\sim p\cap q$. Hence $p-p\cap q\sim r+s_1$, and, since these are disjoint, they are perspective in their union e by Lemma 3. Moreover, since $p\cap q\precsim p'\cap q'$, Lemma 4 implies that $p\cap q$ and $p\cap q+s_2$ are perspective in $f=p\cap q+s_2+p'\cap q'$. Our choice of s is such that e and f are orthogonal, and thus Lemma 2 permits addition of the above perspectivities, thereby giving the perspectivity of p and q.

We proceed similarly in the summand in which $q-p\cap q\precsim p-p\cap q$. In the summand in which $p'\cap q'\precsim p\cap q$ we use the fact that p' and q' are unitarily equivalent to establish as above their perspectivity. But this immediately implies that of p and q, and a final addition of perspectivities via Lemma 2 completes the proof.

COROLLARY. *Perspectivity is transitive in the projection lattice of a w^*-algebra.*

THEOREM 2 (CANTOR-SCHROEDER-BERNSTEIN). *Let L be the projection lattice of a w^*-algebra. If p, $q\in L$ are each perspective to a subprojection of the other, then they are perspective.*

PROOF. By Theorem 1 we need only prove the statement with "perspective" replaced by "unitarily equivalent." We have $p\precsim q$ and $q\precsim p$, and so $p\sim q$. But also p' and q' are each unitarily equivalent to a subprojection of the other, so that $p'\sim q'$. Therefore, p and q are unitarily equivalent.

Our next result is an immediate consequence of Topping's results on weakly closed Jordan algebras of self-adjoint operators [6]. However, in this context, a simpler direct proof is available.

THEOREM 3 (GENERALIZED COMPARABILITY). *Let L be the projection lattice of a w^*-algebra A. For any p, $q\in L$ there is a central projection e of A such that ep is perspective to a subprojection of eq and $e'q$ to a subprojection of $e'p$.*

PROOF. There is a central projection e such that $p-p\cap q\precsim q-p\cap q$ on e and $q-p\cap q\precsim p-p\cap q$ on e'. If $p-p\cap q\sim r\leqq q-p\cap q$ on e then $p-p\cap q$ and r are perspective in their union by Lemma 3. Since this union is orthogonal to $p\cap q$, Lemma 2 implies that p and $p\cap q+r$ are perspective on e. Similarly on e' we have q subperspective to p.

The relation of perspectivity is, in general, not even finitely additive. However, the following improvement of Lemma 2 is valid in any orthomodular lattice. If a_i and b_i are perspective in $[0, e_i]$ for $i \in I$ and the e_i are pairwise orthogonal, then $\sum a_i$ and $\sum b_i$ are perspective.[2] Again, it follows directly from Lemma 2 that if the projections $p = \sum p_i$ and $q = \sum q_i$ are orthogonal and p_i and q_i are perspective for all i, then p and q are perspective.

With regard to possible generalizations, we note that, except for Lemma 4, our arguments are nonspatial, and so apply to AW^*-algebras. More generally, very little seems to be known concerning perspectivity in orthomodular lattices. M. F. Janowitz has called our attention to an example [1, p. 21] of an orthomodular lattice in which perspectivity fails to be transitive.

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[2] This result and Lemma 1 have been obtained independently by S. S. Holland, Jr. in *Distributivity and perspectivity in orthomodular lattice*, Trans. Amer. Math. Soc. **112** (1964), 330–343.