PROOF. If M is Ricci flat it is trivially Einstein and, since $L^2-HL=0$, one of the principal curvatures is zero.

If $R^*=bI$, then $L^2-HL+bI=0$. If $K=0$, then there is a zero principal curvature and a unit principal vector X with $LX=0$. Hence $bX=0$ so $R^*=0$.

In the case $n=3$, the characteristic polynomial $L^3-HL^2+JL-KI=0$ implies $JL=0$, and since $L_m=0$ implies $J(m)=0$, we have $J\equiv 0$.

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UNIVERSITY OF MICHIGAN

ON PSEUDOMETRICS FOR GENERALIZED UNIFORM STRUCTURES

SOLOMON LEADER

In [1] Alfsen and Njåstad generalized the concept of a uniform structure $\mathcal{U}$ on a set S, replacing the intersection axiom for uniform structures by the weaker condition:

(0) Given subsets $A_1, \cdots, A_n$ of S and $U_1, \cdots, U_n$ in $\mathcal{U}$, there exists U in $\mathcal{U}$ such that $U(A_i) \subseteq U_i(A_i)$ for $i=1, \cdots, n$. Our object is to characterize these structures in terms of pseudometrics.

Define a (generalized) *gage* on S to be a nonvoid family $\mathcal{G}$ of pseudometrics on $S\times S$ such that

(1) Every pseudometric uniformly continuous with respect to $\mathcal{G}$ belongs to $\mathcal{G}$.

(2) If α and β belong to $\mathcal{G}$ and both α and β are totally bounded, then $\alpha \vee \beta$ belongs to $\mathcal{G}$.

Note that if we delete the total boundedness condition in (2), then $\mathcal{G}$ is just a gage for a proper uniform structure [2], [3]. For β a pseudometric on $S\times S$, define $W_\beta = \beta^{-1}[0, 1)$.

THEOREM. *Given a gage $\mathcal{G}$ on S, define the class $\mathcal{U}$ of subsets U of $S\times S$ by the condition*

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(i) *$U \in \mathfrak{U}$ iff $U \supseteq W_\beta$ for some β in $\mathcal{G}$.*

Then $\mathfrak{U}$ is a generalized uniform structure for which

(ii) *$\alpha \in \mathcal{G}$ iff $W_{n\alpha} \in \mathfrak{U}$ for every positive integer n.*

Conversely, given a generalized uniform structure $\mathfrak{U}$ on S, define the family $\mathcal{G}$ of pseudometrics α by (ii). *Then $\mathcal{G}$ is a gage for which* (i) *holds.*

LEMMA. *Given a pseudometric α on $S \times S$ and nonempty subsets A and B of S such that $\alpha(A, B) \geqq 1$, there exists a totally bounded pseudometric β such that $\beta \leqq \alpha$ and $\beta(A, B) = 1$.*

To prove the lemma, let $f(s) = \alpha(s, A)[\alpha(s, A) + \alpha(s, B)]^{-1}$ on S and define $\beta(x, y) = |f(x) - f(y)|$. A few simple computations show that β has the desired properties.

To prove the theorem, let $\mathcal{G}$ satisfy (1) and (2) and define $\mathfrak{U}$ by (i). That $\mathfrak{U}$ has all the properties of a uniform structure except for the intersection axiom follows exactly as in the case of proper uniform structures. To prove (0) we may, in view of (i), assume that $U_i = W_\alpha$ for some $\alpha_i = \alpha$ in $\mathcal{G}$. Let $B_i = S \backsim U_i(A_i)$. Since the conclusion of (0) will be trivial wherever A_i or B_i is empty, we may assume both are nonempty. Apply the lemma to get β_i totally bounded with $\beta_i \leqq \alpha_i$ and $\beta_i(A_i, B_i) = 1$. β_i is in $\mathcal{G}$ by (1). Let $\beta = \beta_1 \vee \cdots \vee \beta_n$, which is in $\mathcal{G}$ by (2). Let $U = W_\beta$. Given y in $U(A_i)$, (x, y) is in U for some x in A_i. That is, $\beta_i(x, y) \leqq \beta(x, y) < 1$ for some x in A_i. So $\beta_i(y, A_i) < 1$. Since $\beta_i(A_i, B_i) = 1$, y is not in B_i. That is, y is in $U_i(A_i)$. Thus (0) holds and $\mathfrak{U}$ is a generalized uniform structure.

To prove (ii), consider any α in $\mathcal{G}$. Then $n\alpha$ is in $\mathcal{G}$ by (1) and hence $W_{n\alpha}$ is in $\mathfrak{U}$ by (i). Conversely, let $W_{n\alpha}$ belong to $\mathfrak{U}$ for all n. By (i) there exists for each n some β in $\mathcal{G}$ such that $W_\beta \subseteq W_{n\alpha}$. Thus (1) implies α is in $\mathcal{G}$.

Given a generalized uniform structure $\mathfrak{U}$, define $\mathcal{G}$ by (ii). We must prove (1), (2), and (i). For β uniformly continuous relative to $\mathcal{G}$ and m any positive integer, there exist α in $\mathcal{G}$ and a positive integer n such that $W_{n\alpha} \subseteq W_{m\beta}$. Since $W_{n\alpha}$ belongs to $\mathfrak{U}$ by (ii), so does $W_{m\beta}$. So β is in $\mathcal{G}$ by (ii). Hence (1) holds just as in the case of proper uniform structures.

To prove (2) let α and β be totally bounded members of $\mathcal{G}$. Let $\gamma = \alpha \vee \beta$. Since γ is totally bounded, we can get a finite covering $S_1 \cup \cdots \cup S_k = S$ with diameters $\gamma[S_i] < 1/4$. Applying (0) to the sequences

$$\left\{ \begin{matrix} S_1, \cdots, S_k, S_1, \cdots, S_k \\ W_{2\alpha}, \cdots, W_{2\alpha}, W_{2\beta}, \cdots, W_{2\beta} \end{matrix} \right\}$$

we get U in $\mathfrak{U}$ such that

$$(3) \qquad U(S_i) \subseteq W_{2\alpha}(S_i) \cap W_{2\beta}(S_i) \quad \text{for } i = 1, \cdots, k.$$

Consider any (x, y) in U. Since x is in some S_i, y is in the corresponding $U(S_i)$. Hence (3) implies $\gamma(y, S_i) < 3/4$. So $\gamma(x, y) \leqq \gamma(x, S_i) + \gamma[S_i] + \gamma(y, S_i) < 0 + 1/4 + 3/4 = 1$. That is, $U \subseteq W_\gamma$. So W_γ belongs to $\mathfrak{U}$ whenever α and β are totally bounded members of $\mathcal{G}$. Using (1) we can apply this result to $n\alpha$ and $n\beta$ to conclude that $W_{n\gamma}$ belongs to $\mathfrak{U}$. That is, γ is in $\mathcal{G}$. So (2) holds.

To prove (i) let U be any member of $\mathfrak{U}$. Choose a sequence $\{U_n\}$ in $\mathfrak{U}$ such that $U_n = U_n^{-1}$ and $U_{n+1}^3 \subseteq U_n \subseteq U$ for all n. By the Metrization Lemma [3] there exists a pseudometric β such that

$$(4) \qquad U_{n+1} \subseteq W_{2^{n-1}\beta} \subseteq U_n \qquad \text{for all } n.$$

β is in $\mathcal{G}$ by (4) and (ii). Setting $n = 1$ in (4) yields $W_\beta \subseteq U$ which proves the direct implication in (i). The converse follows from (ii) since W_β is in $\mathfrak{U}$ if β is in $\mathcal{G}$.

Using the lemma and [4], we obtain the following corollaries.

COROLLARY 1. *For a given proximity relation, let $\mathfrak{J}$ be the associated precompact gage and $\mathcal{G}$ be the associated total* [1] *gage. Then $\mathcal{G}$ consists of all pseudometrics α on $S \times S$ such that every totally bounded pseudometric β satisfying $\beta \leqq \alpha$ belongs to $\mathfrak{J}$.*

COROLLARY 2. *A gage $\mathcal{G}$ is total iff $\mathcal{G}$ contains every pseudometric α for which every totally bounded pseudometric β satisfying $\beta \leqq \alpha$ belongs to $\mathcal{G}$.*

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RUTGERS, THE STATE UNIVERSITY