A NOTE ON COUNTING ISOTROPY SUBGROUPS

L. N. MANN

1. **Introduction.** An *elementary p-group of rank k* is a group isomorphic to the direct sum of k copies of Z_p, the additive group of integers modulo a prime p. P. A. Smith has investigated the actions of such a group G of homeomorphisms on the n-sphere S^n and has observed certain similarities with the standard orthogonal actions. In particular, he has shown [4], [5] that if G acts effectively on S^n, then $k = \text{rank } G \leqq [(n+1)/2]$ for p odd and $k \leqq n+1$ for $p=2$. As usual, $[x]$ denotes the largest integer not exceeding x. The purpose of this note is to show that if such a group G acts effectively on S^n, then the number of distinct isotropy subgroups cannot exceed $2^{[(n+1)/2]}$ for p odd, and $2^{n+1}-1$ for $p=2$. These are precisely the bounds which exist for orthogonal actions.

The proof proceeds by first observing that every *maximal* isotropy subgroup is of rank $k-1$. A formula of Borel [1] is then utilized to show that the number of maximal isotropy subgroups cannot exceed $[(n+1)/2]$ for p odd, $n+1$ for $p=2$, and that, moreover, each isotropy subgroup of rank $k-i$, $1 \leqq i \leqq k-1$, is the intersection of i maximal isotropy subgroups. Noting that $k \leqq [(n+1)/2]$ for p odd and $k \leqq n+1$ for $p=2$, and allowing for the isotropy subgroups G and $\{e\}$, the result follows.

2. **Definitions and preliminaries.** Given an action of a topological transformation group G on a space X, the *isotropy subgroup at a point* $x_0 \in X$, denoted by G_{x_0}, is defined as the subgroup of all elements of G which leave x_0 fixed. The action is said to be *effective* if $\bigcap_{x\in X} G_x = e$, the identity element of G; it is said to be *free* if $\{e\}$ is the only isotropy subgroup. The fixed-point set of the action, denoted by $F(G, X)$, is the subset of X of all points with isotropy subgroup G.

All spaces considered will be compact Hausdorff spaces and the usual Čech cohomology will be used. A *cohomology n-sphere over Z_p* is a space with the cohomology groups, coefficient group Z_p, of S^n. A *generalized cohomology n-sphere over Z_p* is a cohomology n-sphere over Z_p which is also a cohomology n-manifold over Z_p [1]. Results of Smith [2], [3], [5] show that if an elementary p-group G acts on a cohomology n-sphere (generalized cohomology n-sphere) X over Z_p, then $F(G, X)$ is a cohomology r-sphere (generalized cohomology

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r-sphere) over Z_p, $r \leqq n$. Moreover, if X is a generalized cohomology n-sphere and the action is effective, then $r \leqq n-2$ and $n-r$ is even for p odd, and $r \leqq n-1$ for $p=2$. We shall use strongly the following result of Borel [1]. (For $F(K, X)$ empty, it is agreed that $n(K, X) = -1$.)

THEOREM 1 (BOREL). *Suppose G is an elementary p-group acting effectively on a cohomology n-sphere X over Z_p. For each subgroup K of G let $n(K, X)$ be the integer such that $F(K, X)$ is a cohomology $n(K, X)$-sphere over Z_p. Then*

$$n - n(G, X) = \sum_H (n(H, X) - n(G, X)),$$

where H runs through the subgroups of index p in G.

It will prove useful to have available the following group-theoretic result which we present without proof.

LEMMA 1. *Let G be an elementary p-group of rank k and K a subgroup of G of rank $k-i$, $1 \leqq i \leqq k-1$. Suppose K is the intersection of t distinct subgroups of G, each of rank $k-1$. Then $t \geqq i$ and K is the intersection of some subcollection of i of the t subgroups.*

3. Main results.

THEOREM 2. *Suppose G is an elementary p-group acting effectively on a generalized cohomology n-sphere X over Z_p. Let $r = n(G, X) \geqq -1$. Then the number of distinct isotropy subgroups cannot exceed $2^{(n-r)/2}$ for p odd or 2^{n-r} for $p=2$.*

PROOF. Suppose rank $G = \Gamma(G) = k$. We shall say that an isotropy subgroup of rank less than k is *maximal* if it is not properly contained in any isotropy subgroup with the possible exception of G. We first show that each maximal isotropy subgroup is of rank $k-1$. Suppose, on the contrary, that S is a maximal isotropy subgroup with $\Gamma(S) \leqq k-2$. Now G/S leaves the generalized cohomology sphere $F(S, X)$ invariant. Moreover, since S is a maximal isotropy subgroup, it is easy to see that G/S acts freely outside of the fixed-point set on $F(S, X)$. In fact, $F(G/S, F(S, X)) = F(G, X)$. Now $F(S, X) \supset F(G, X) = F(G/S, F(S, X))$ and the action of G/S on $F(S, X)$ is effective. We apply Theorem 1 to this action since $\Gamma(G/S) \geqq 2$. We have

$$n(S, X) - n(G, X) = \sum_H (n(H, F(S, X)) - n(G, X)), \tag{1}$$

where H runs through the subgroups of index p in G/S. Recalling that G/S acts freely outside of $F(G, X)$ on $F(S, X)$, it follows that

$F(H, F(S, X))=F(G, X)$ for all H and, therefore, the right-hand side of (1) is zero. However, since G/S acts effectively on $F(S, X)$, it follows that the left-hand side of (1) is strictly positive, giving us a contradiction. Hence, each maximal isotropy subgroup is of rank $k-1$. This conclusion could also have been obtained by using the results of [4].

Next we observe that there are at most $(n-r)/2$ maximal isotropy subgroups for p odd. But this follows immediately from Theorem 1 since each term $(n(H, X)-n(G, X))$ must be even for an odd prime. For $p=2$, we conclude that there are at most $n-r$ maximal isotropy subgroups.

Suppose that $T\subset S\subset G$, $\Gamma(S)=k-1$, and $\Gamma(T)=k-2$, with S an isotropy subgroup of the action of G on X and T an isotropy subgroup of the action of S on X. We wish to conclude that T is also an isotropy subgroup of the action of G on X. We know that $F(T, X)\supset F(S, X)$. Suppose T is not an isotropy subgroup of G on X. Then

$$F(T, X) \subseteq \bigcup_i F(S_i, X),$$

for some collection of isotropy subgroups S_i of G on X where $S_i\supset T$ and $\Gamma(S_i)=k-1$ for each i. Since $S_i\supset T$, we have $F(T, X)\supseteq F(S_i, X)$ and

$$F(T, X) = \bigcup_i F(S_i, X).$$

Due to dimensional restrictions (we are dealing with connected cohomology manifolds), we must have $F(T, X)=F(S_{i_0}, X)$ for some i_0. Hence, $F(S_{i_0}, X)=F(T, X)\supset F(S, X)$ which contradicts S being an isotropy subgroup.

We now come to the crux of the argument: to show that if R is an isotropy subgroup of G of rank $k-i$, $1\leqq i\leqq k-1$, then R is the intersection of i distinct isotropy subgroups of rank $k-1$. By Lemma 1, it is sufficient to show that R is the intersection of some collection of isotropy subgroups of rank $k-1$. We consider first the case that $\Gamma(k)=k-2$. It is sufficient to exhibit two distinct maximal isotropy subgroups which contain R. Consider the action of G/R, $\Gamma(G/R)=2$, on the generalized cohomology sphere $F(R, X)$. We have $F(G/R, F(R, X))=F(G, X)\subset F(R, X)$. As R is an isotropy subgroup, this action must be effective. Applying Theorem 1 to the action, one sees that there must exist distinct cyclic subgroups K_1^* and K_2^* of G/R with $F(K_j^*, F(R, X))\supset F(G, X)$ for $j=1, 2$. Since K_1^* and K_2^* generate G/R, $F(K_1^*, F(R, X))\neq F(K_2^*, F(R, X))$; for, otherwise, $F(G/R, F(R, X))$ would be $F(K_1^*, F(R, X))$ instead of $F(G, X)$. Let π be the projection $\pi: G \to G/R$, and let $S_j = \pi^{-1}(K_j^*)$, $j=1, 2$. We claim $F(S_1, X) \neq F(S_2, X)$ and $F(S_j, X) \supset F(G, X)$ for $j=1, 2$. To see this observe that $F(S_j, X) = F(S_j/R, F(R, X)) = F(K_j^*, F(R, X))$. Now suppose that $R = G_{x_0}$. Then there exists $y_j \in F(S_j, X)$ with $y_j \neq x_0$ and $y_j \notin F(G, X)$, $j=1, 2$; and, moreover, $y_1 \neq y_2$. It follows that $S_j = G_{y_j}$, $j=1, 2$, and we have two distinct maximal isotropy subgroups, S_1 and S_2, containing R.

We proceed by induction on $k=\Gamma(G)$, starting with $k=3$. But if $\Gamma(G)=3$, we need consider isotropy subgroups R only of rank $k-2$ (that is, of rank 1), and the argument above immediately applies. Suppose then that R is an isotropy subgroup of G, $\Gamma(G)=k$. Then there exists a maximal isotropy subgroup S of G with $R \subset S$ and $\Gamma(S)=k-1$. Consider the action of S on X. By our induction hypothesis, R is the intersection of a collection T_j of isotropy subgroups of S of rank $k-2$. By an argument above, each T_j is also an isotropy subgroup of the action of G on X and, therefore, the intersection of two maximal isotropy subgroups of rank $k-1$. Finally, then, R is the intersection of a collection of isotropy subgroups, each of rank $k-1$.

We now know that there are at most binomial coefficient $\binom{(n-r)/2}{i}$ distinct isotropy subgroups of rank $k-i$, $1 \leqq i \leqq k-1$, for p odd; at most $\binom{n-r}{i}$ distinct ones for $p=2$. Noting that $k \leqq (n-r)/2$ for p odd and $k \leqq n-r$ for $p=2$, an immediate generalization of Smith's result in [4], [5], and allowing for the isotropy subgroups G and $\{e\}$ of rank k and 0, respectively, the theorem follows. Actually, if $r=-1$, that is, $F(G, X)$ is empty, then we may omit G as a possible isotropy subgroup and we observe that there exist at most $2^{(n+1)/2}-1$ distinct isotropy subgroups for p odd, and at most $2^{n+1}-1$ for $p=2$.

COROLLARY. *Let G be an elementary p-group acting effectively on E^n, euclidean n-space. Then the number of distinct isotropy subgroups cannot exceed $2^{[n/2]}$ for p odd and 2^n for $p=2$.*

PROOF. Extend the action of G to S^n by leaving the point at infinity fixed. Of course, a stronger statement of the Corollary is possible in terms of the cohomology dimension, r, of $F(G, E^n)$.

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Institute for Defense Analyses and
University of Virginia

DIFFERENTIABLE ACTIONS OF COMPACT ABELIAN LIE GROUPS ON S^n

L. N. MANN

1. **Introduction.** In [9] P. A. Smith raises the following question: If a finite group G acts effectively on the n-sphere S^n, must there also be some effective orthogonal action of G on S^n? Stated another way, must all finite groups acting effectively on S^n be isomorphic to subgroups of the orthogonal group $O(n+1)$? Smith has answered this question in the affirmative for the case where G is an elementary p-group [8], [9]. The Corollary to Theorem 2 of this paper settles the case where G is a compact abelian Lie group (in particular, a finite abelian group) and the action is assumed differentiable.

The proof of our main result is immediate if one assumes the existence of a fixed point, as evidenced by the following result which utilizes Bochner's theorem on local linearity about a fixed point.

Theorem 1. *Let G be a compact Lie group operating effectively and differentiably on a differentiable n-manifold X. If there exits a point x_0 left fixed by every element of G, then G is isomorphic to a subgroup of $O(n)$.*

Proof. By Bochner's theorem [5, p. 206], we may assume G acts orthogonally (but not necessarily effectively) on some small closed n-cell D with center x_0. G leaves bdy $D=S^{n-1}$ invariant. If G is not effective on S^{n-1}, then there must be a homeomorphism g_0 of finite order in G which leaves S^{n-1} pointwise fixed. Since g_0 acts linearly on D and leaves x_0 fixed, it must also leave D pointwise fixed. By Newman's theorem [5, p. 223], g_0 must leave X pointwise fixed, violating the effectiveness of G on X. Hence G acts orthogonally and effectively on S^{n-1}, and the theorem follows.

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