SOME PROPERTIES OF CERTAIN SETS OF COPRIME INTEGERS

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1. Introduction. The set $P(n)$ of all primes less than or equal to n has the obvious property that it contains exactly one multiple of each prime less than or equal to n. We use this partial description of $P(n)$ as a basis for the following

DEFINITION 1.1. An increasing sequence $\{a_1, \cdots, a_k\}$ of integers greater than 1 is a *coprime chain* iff it contains exactly one multiple of each prime equal to or less than a_k.

The following is a list of all coprime chains $\{a_1, \cdots, a_k\}$ with $a_k \leqq 13$.

$\{2\}$;
$\{2,3\}$;
$\{3,4\}$;
$\{2,3,5\}$, $\{3,4,5\}$;
$\{5,6\}$;
$\{2,3,5,7\}$, $\{3,4,5,7\}$, $\{5,6,7\}$;
$\{3,5,7,8\}$;
$\{2,5,7,9\}$, $\{4,5,7,9\}$, $\{5,7,8,9\}$;
$\{3,7,10\}$, $\{7,9,10\}$;
$\{2,3,5,7,11\}$, $\{3,4,5,7,11\}$, $\{3,5,7,8,11\}$, $\{2,5,7,9,11\}$, $\{4,5,7,9,11\}$;
$\{5,7,8,9,11\}$, $\{5,6,7,11\}$, $\{3,7,10,11\}$, $\{7,9,10,11\}$;
$\{5,7,11,12\}$;
$\{2,3,5,7,11,13\}$, $\{3,4,5,7,11,13\}$, $\{3,5,7,8,11,13\}$, $\{2,5,7,9,11,13\}$;
$\{4,5,7,9,11,13\}$, $\{5,7,8,9,11,13\}$, $\{5,6,7,11,13\}$, $\{3,7,10,11,13\}$;
$\{7,9,10,11,13\}$, $\{5,7,11,12,13\}$.

The notation $A(n)$, $B(n)$, etc., will be used to designate coprime chains having n as largest member.

In this paper we are mainly concerned with finding functions asymptotic to the sum of the rth powers of the members of a coprime chain $A(n)$. A later paper will deal with the number of coprime chains with largest member n.

All assumed results are well known and can be found in [1].

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2. **The sum of powers of members of a coprime chain.** The brief table of coprime chains given in the introduction suggests that a coprime chain $A(n)$ has, asymptotically, $\pi(n)$ members. We will prove a stronger theorem but we first require the

LEMMA 2.1.

$$\sum_{p \leq n} p^r \sim \frac{n^{r+1}}{(r+1)\log n}, \qquad r > -1.$$

PROOF. We note that for $r=0$ this result is just the prime number theorem. For $r>-1$ partial summation gives

$$\sum_{p \leq n} p^r = \pi(n)n^r - r\int_2^n \pi(t)t^{r-1}\,dt.$$

But, since

$$\int_2^n \pi(t)t^{r-1}\,dt \sim \int_2^n t^r\left(\frac{1}{\log t} - \frac{1}{(r+1)\log^2 t}\right)dt \sim \frac{n^{r+1}}{(r+1)\log n},$$

we have the desired result.

THEOREM 2.2.

$$\sum_{a \in A(n)} a^r = \sum_{p \leq n} p^r + O\left(\frac{n^{(cr+1)/2}}{\log n}\right),$$

where $c=1$ *for* $-1<r\leqq 0$ *and* $c=2$ *for* $r>0$.

PROOF. Partition $A(n)$ into the sets $P=\{p\in A(n) \mid p \text{ is prime}\}$ and $M=A(n)-P$. Then

$$\sum_{a \in A(n)} a^r = \sum_{p \leq n} p^r - \sum_{p \leq n;\, p \notin P} p^r + \sum_{m \in M} m^r.$$

Each member of M is divisible by some prime less than or equal to $\sqrt{n}$ and, since the members of M are coprime in pairs, M has at most $\pi(\sqrt{n})$ members.

(i) Assume $-1<r\leqq 0$. For each m in M choose a prime divisor q of m. Then

$$\sum_{m \in M} m^r \leqq \sum_{m \in M} q^r \leqq \sum_{p \leq \sqrt{n}} p^r = O\left(\frac{n^{(r+1)/2}}{\log n}\right)$$

by Lemma 2.1.

To complete the proof of this part of the theorem we note that there are at most $\pi(\sqrt{n})$ primes p satisfying $\sqrt{n}<p\leqq n$, $p\notin P$. Thus

$$\sum_{p\leq n;\, p\notin P} p^r \leqq \sum_{p\leq\sqrt{n}} p^r + \sum_{\sqrt{n}<p\leq n;\, p\notin P} p^r = O\left(\frac{n^{(r+1)/2}}{\log n}\right).$$

(ii) Assume $r>0$. Choose $n_0>2^{2/r}$. Then, for $n>n_0$,

$$\sum_{p\leq n} p^r - \sum_{p\leq\sqrt{n}} p^r = \sum_{\sqrt{n}<p\leq n} p^r \leqq \sum_{a\in A(n)} a^r \leqq \sum_{p\leq n} p^r + \sum_{m\in M} m^r.$$

Now

$$\sum_{m\in M} m^r \leqq n^r\pi(\sqrt{n}) = O\left(\frac{n^{r+1/2}}{\log n}\right)$$

and

$$\sum_{p\leq\sqrt{n}} p^r = O\left(\frac{n^{(r+1)/2}}{\log n}\right), \quad \text{which gives} \quad \sum_{p\leq\sqrt{n}} p^r = O\left(\frac{n^{r+1/2}}{\log n}\right).$$

This completes the proof of the theorem.

Applying Lemma 2.1 to the above result we obtain the

COROLLARY 2.3.

$$\sum_{a\in A(n)} a^r \sim \frac{n^{r+1}}{(r+1)\log n}, \qquad r>-1.$$

As the next theorem shows, Theorem 2.2 is the best possible, in that no error term of lower order will suffice.

THEOREM 2.4. *For all sufficiently large n, there exist coprime chains $A(n)$ and $B(n)$ such that*

$$\left|\sum_{a\in A(n)} a^r - \sum_{b\in B(n)} b^r\right| \geqq c_1 \frac{n^{(cr+1)/2}}{\log n},$$

where c is defined as in Theorem 2.2 and c_1 is a constant depending on r only and is positive for $-1<r\neq 0$.

PROOF. Let $\{q_1, \cdots, q_{k-1}\}$ be the set of primes less than n and not dividing n. Let $a_i=q_i^{[\log n/\log q_i]}$, $b_i=q_i$, $i=1, \cdots, k-1$, $a_k=b_k=n$. Then $A(n)=\{a_1, \cdots, a_k\}$ and $B=\{b_1, \cdots, b_k\}$ are both coprime chains.

(i) Assume $-1<r<0$. Then

$$\left|\sum_{a\in A(n)} a^r - \sum_{b\in B(n)} b^r\right|$$

$$= \sum_{p<n} p^r - \sum_{p|n} p^r - \sum_{p<n} p^{r[\log n/\log p]} + \sum_{p|n} p^{r[\log n/\log p]}$$

$$\geqq \sum_{p\leqq\sqrt{n}} p^r - \sum_{p|n} p^r - \sum_{p\leqq\sqrt{n}} p^{r[\log n/\log p]}$$

$$\geqq \frac{\{1+o(1)\}n^{(r+1)/2}}{\frac{r+1}{2}\log n} - \sum_{p|n} 1 - \sum_{p\leqq\sqrt{n}} p^{r(\log n/\log p-1)}$$

$$\geqq \frac{2\{1+o(1)\}}{r+1}\frac{n^{(r+1)/2}}{\log n} - \sum_{p\leqq\sqrt{n}} n^r p^{-r} = \{1+o(1)\}\frac{-4r}{1-r^2}\frac{n^{(r+1)/2}}{\log n}.$$

(ii) Assume $r>0$. Then

$$\left|\sum_{a\in A(n)} a^r - \sum_{b\in B(n)} b^r\right| \geqq \sum_{p\leqq\sqrt{n}} p^{r[\log n/\log p]} - \sum_{p|n} p^{r[\log n/\log p]} - \sum_{p\leqq\sqrt{n}} p^r$$

$$\geqq \sum_{p\leqq\sqrt{n}} p^{2r} - \sum_{p|n} n^r - O\left(\frac{n^{(r+1)/2}}{\log n}\right)$$

$$\geqq \{1+o(1)\}\frac{n^{r+1/2}}{(r+\frac{1}{2})\log n} + O(n^r\log n) + O(n^{(r+1)/2})$$

$$= \{1+o(1)\}\frac{2}{2r+1}\frac{n^{r+1/2}}{\log n}.$$

The above theorem is also valid for $r=0$, as will be shown in Theorem 3.5.

The first major difference between coprime chains and sets of consecutive primes becomes apparent in the following

THEOREM 2.5. *If, for each n, coprime chains $A(n)$ and $B(n)$ are chosen so that $\sum_{a\in A(n)} 1/a$ and $\sum_{b\in B(n)} 1/b$ are maximal and minimal, respectively, then*

$$\sum_{a\in A(n)} \frac{1}{a} \sim \log\log n \quad \textit{and} \quad \sum_{b\in B(n)} \frac{1}{b} \to \log 2 \quad \textit{as} \quad n\to\infty.$$

PROOF. Clearly $A(n)-\{n\}$ is the set of primes less than n that do not divide n. Hence

$$\sum_{a\in A(n)} \frac{1}{a} = \sum_{p\leq n} \frac{1}{p} - \sum_{p\mid n} \frac{1}{p} + \frac{1}{n} = \log\log n - \sum_{p\mid n} \frac{1}{p} + O(1).$$

Since n has no more than 2 log n distinct prime divisors, it follows that

$$\sum_{p\mid n} \frac{1}{p} = O(\log\log\log^2 n),$$

for all sufficiently large n. Thus

$$\sum_{a\in A(n)} \frac{1}{a} \sim \log\log n.$$

To complete the proof we now consider any coprime chain $B(n)$ chosen so that $\sum_{b\in B(n)} 1/b$ is minimal for fixed n and note that $B(n)$ can contain no number less than or equal to $\sqrt{n}$. We define P and M as in the proof of Theorem 2.2. Then

$$\begin{aligned}\sum_{b\in B(n)} \frac{1}{b} &= \sum_{p\in P} \frac{1}{p} + \sum_{m\in M} \frac{1}{m} \\ &= \sum_{\sqrt{n}<p\leq n} \frac{1}{p} - \sum_{\sqrt{n}<p\leq n;\, p\notin P} \frac{1}{p} + O\left(\sum_{m\in M} \frac{1}{\sqrt{n}}\right) \\ &= \log 2 - \sum_{\sqrt{n}<p\leq n;\, p\notin P} \frac{1}{p} + O\left(\frac{\pi(\sqrt{n})}{\sqrt{n}}\right) \\ &= \log 2 - \sum_{\sqrt{n}<p\leq n;\, p\notin P} \frac{1}{p} + o(1).\end{aligned}$$

Again, using the fact that M has at most $\pi(\sqrt{n})$ members, we have

$$\sum_{\sqrt{n}<p\leq n;\, p\notin P} \frac{1}{p} = O\left(\frac{\pi(\sqrt{n})}{\sqrt{n}}\right) = o(1).$$

Hence

$$\sum_{b\in B(n)} \frac{1}{b} \to \log 2 \quad \text{as} \quad n\to\infty,$$

and the proof of the theorem is complete.

If $\{A(n)\}$ is any sequence of coprime chains, then the sequence whose members are $\sum_{a\in A(n)} a^r$ is bounded for $r<-1$, but for certain sequences $\{B(n)\}$ we may obtain a more precise result.

THEOREM 2.6. *There exists a sequence $\{B(n)\}$ of coprime chains* *such that the sequence whose members are* $\sum_{b\in B(n)} b^r$ *converges to* 0 *for all* $r<-1$.

PROOF. Assume $r<-1$, $0<\epsilon<1$ given. Choose n_0 so that $n_0^{(r+1)/2}<\epsilon/\log 2$. Let $\{B(n)\}$ be a sequence of coprime chains chosen so that, for fixed n, $\sum_{b\in B(n)} 1/b$ is minimal. By Theorem 2.5 there is an n_1 so that $\sum_{b\in B(n)} 1/b<\epsilon+\log 2$ for all $n>n_1$. Since $B(n)$ contains no number less than or equal to $\sqrt{n}$ we have $b^{1+r}<n^{(r+1)/2}$ for all b in $B(n)$. Thus, for all $n>n_0n_1$, we have $b^r<(1/b)\epsilon/\log 2$ for each b in $B(n)$ and, hence,

$$0<\sum_{b\in B(n)} b^r<\frac{\epsilon}{\log 2}\sum_{b\in B(n)}\frac{1}{b}<\epsilon+\frac{\epsilon^2}{\log 2}<3\epsilon,$$

and the proof is complete.

3. Coprime chains of maximal and minimal length.

DEFINITION 3.1. For each $n>1$ choose coprime chains $A(n)$ and $B(n)$ so that $\sum_{a\in A(n)} 1$ and $\sum_{b\in B(n)} 1$ are maximal and minimal, respectively. Define

$$m(n)=\sum_{a\in A(n)} 1 \quad\text{and}\quad l(n)=\sum_{b\in B(n)} 1.$$

Now $m(n)$ and $l(n)$ are about the same size; more precisely, setting $r=0$ in Corollary 2.3 gives $m(n)\sim l(n)\sim n/\log n$. However, we can make more precise statements about both $m(n)$ and $l(n)$.

THEOREM 3.2. $l(n)$ *assumes every positive integral value.*

PROOF. From the table in the first section, $l(2)=1$ and, since $l(n)\to\infty$, it suffices to show $l(n+1)\leqq l(n)+1$, $n>1$.

Let $\{a_1,\cdots,a_k=n\}$ be a coprime chain of minimal length with a_i, $i=1,\cdots,k-1$, square-free. Let $b_i=a_i/(a_i,n+1)$, $i=1,\cdots,k-1$. Then the members of $B(n+1)=\{b_1,\cdots,b_{k-1},n,n+1\}$ are relatively prime in pairs and every prime less than or equal to $n+1$ divides some member of $B(n+1)$. Thus, if all 1's are deleted from $B(n+1)$ and the remaining members are properly reordered, we obtain a coprime chain. Then $l(n+1)\leqq k+1=l(n)+1$ and the proof may be completed by induction.

THEOREM 3.3. $m(n)=\pi(n)-\omega(n)+1$, *where* $\omega(n)$ *is the number of different prime factors of* n.

PROOF. Clearly a coprime chain of maximal length can be constructed by using only n and all primes less than and relatively prime to n.

COROLLARY 3.4. *$m(n)$ assumes every positive integral value.*

PROOF. Letting $n=p_k$ in the previous theorem we obtain $m(p_k)=k$. In Theorem 2.4 the restriction $r\neq 0$ is unnecessary in view of the following

THEOREM 3.5. *Given $\epsilon>0$, $m(n)-l(n)>(1-\epsilon)\sqrt{n}/\log n$ for all sufficiently large n.*

PROOF. If $\{a_1, a_2, \cdots, a_{2k-1}, a_{2k}, a_{2k+1}, \cdots, n\}$ is a coprime chain and $a_{2k}<\sqrt{n}$, then $\{a_1a_2, a_3a_4, \cdots, a_{2k-1}a_{2k}, a_{2k+1}, \cdots, n\}$ can be reordered to form a coprime chain. Now the coprime chain containing n as largest member and all primes less than and relatively prime to n contains at least $\pi(\sqrt{n})-\omega(n)-1$ members less than $\sqrt{n}$. By pairing these members as indicated above we can form a coprime chain with at most $m(n)-\frac{1}{2}[\pi(\sqrt{n})-\omega(n)-1]$ members. But since $\omega(n)<2\log n$ we have

$$l(n) \leqq m(n) - \frac{1}{2}\left[\pi(\sqrt{n}) - 2\log n - 1\right] = m(n) - \frac{\sqrt{n}}{\log n}\{1 + o(1)\}.$$

REFERENCE

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 3rd ed., Oxford Univ. Press, New York, 1954.

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