UPPER SEMICONTINUOUS DECOMPOSITIONS OF DEVELOPABLE SPACES[1]

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Presented here are theorems concerning upper semicontinuous decompositions of developable spaces, topological in the sense that the common parts of intersecting domains (open sets) are open. Theorem 1 shows that, if the elements of such a decomposition do not have nonbicompact [1] intersections with the closures of their complements, the decomposition space is developable. Theorem 2 shows that if additionally the space covered by the decomposition is complete in a certain Cauchy sense defined below the decomposition space is complete in this sense. Theorem 3 is a variation of Theorem 2 dealing with a nonequivalent [10] Ascoli type completeness property. Under Consequences some implications of Theorems 1 and 2 are given. One of these gives affirmative resolutions of the following questions raised by R. L. Moore: (1) *Do upper semicontinuous decompositions into compact point sets of spaces satisfying Axiom* 0 *and Axiom* 1_3 (*the first three conditions of Axiom* 1) *of "Foundations of point set theory"* [6] *yield spaces satisfying these axioms?* (2) *Do such decompositions of spaces satisfying Axiom* 1 *yield spaces satisfying this axiom?* Two other consequences are theorems of Morita-Hanai-Stone [7], [12] and I. A. Vaĭnšteĭn [13].

The sequence G_1, G_2, G_3, $\cdots$ is said to be a *development* of the space Σ provided that (1) for each n, G_n is a collection of domains covering Σ and (2) if P is a point and D is a domain containing P, then for some n every element of G_n containing P is a subset of D. A space is said to be *developable* provided it has a development [2]. The developable space Σ is said to be complete in sense C or sense A accordingly as it has a decreasingly monotonic development G_1, G_2, G_3, $\cdots$ satisfying the first or second of the following conditions: *Condition* C. If J is an infinite point set and for each n some g_n of G_n contains all except finitely many points of J, then there exists a point P such that every domain containing P contains infinitely many

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points of J. *Condition* A. If J and the g_n's are as above and each g_{n+1} is a subset of g_n, there exists a point P as above.

The decomposition G of the topological space Σ is called *upper semicontinuous* in Stone's article cited above provided that for each g in G every domain containing g has a subdomain D containing g such that D contains every element of G that it intersects. Stone calls attention to a variation in meaning for this terminology in the literature. The above definition does not require that the decomposition's elements be closed but, if they are, reduces to the definition in [6] for the spaces under consideration here.

Throughout the remainder of this treatment, Σ denotes a developable topological space; G denotes an upper semicontinuous decomposition of Σ, in the above sense, no element of which has a nonbicompact intersection with the closure of its complement; and I denotes the space in which points are the elements of G and regions are the subcollections of G the sums of the elements of which are domains in Σ. Moore's Axiom 0, which states that every region is a point set, is assumed to hold true in Σ with *domain*, *closure*, etc., being defined naturally in terms of the notion of region. Following Moore's usage, the notation K^* denotes the sum of the sets of the collection K. As in [6], no empty set is used. Suitable adjustments in cited definitions may be made where needed on this account.

THEOREM 1. *I is developable.*

PROOF. Σ has a decreasingly monotonic development. Let G_1, G_2, $G_3, \cdots$ denote one satisfying Condition C if Σ is complete in sense C and satisfying Condition A if Σ is complete in sense A. It may be shown that there exist meanings for the notations $M_{k,h}$ and $U_{k,h}$ for positive integers k and regions h in I such that with respect to a sequence H_1, H_2, $H_3, \cdots$ of well-ordered collections of regions covering I these conditions hold true: (1) For each k and region h of H_k, $M_{k,h}$ is a point of h but of no preceding region and $U_{k,h}$ is a collection of elements of G_k intersecting $M_{k,h}$. Moreover, if $M_{k,h}$ is not a domain in Σ then $U_{k,h}$ is finite and covers both $h^*-M_{k,h}$ and β, the set of all points belonging to $M_{k,h}$ and the boundary of $M_{k,h}$. But if $M_{k,h}$ is a domain it is h^*. (2) If k, h, and β are as above, each member u of $U_{k,h}$ contains an element P of β such that if $n<k$ and h' is the first region of H_n containing $M_{k,h}$ then u is a subset of every member of $U_{n,h'}$ that contains P. (3) If $n<k$ and g belongs to G then the first region of H_k containing g is a subset of the first region of H_n doing so.

If A belongs to the region R in I and is a domain in Σ there exist an teger s and element P of A such that every member of G_s containing

P is a subset of A. If $i \geqq s$ and the region h of H_i contains A then A is $M_{i,h}$, for if it is not then some element of $U_{i,h}$ contains P and is therefore a subset of A intersecting $M_{i,h}$. Hence h^* is A and h is a subset of R. If A is not a domain, then for each n there exists a first region h_n of H_n containing A. Let T_n denote U_{n,h_n} and let C_n denote M_{n,h_n}. By condition (1) above, the collections T_n are finite and cover β, the intersection of A and the closure of the complement of A. Thus for some n there exists a subcollection V of $T_1 + \cdots + T_n$ covering the bicompact point set β such that V^* is a subset of R^*. Let Y denote a point of β and let R' denote a region in I containing A such that R'^* is a subset of $V^* + A$. There exists some $k > n$ such that every element of G_k containing Y is a subset of R'^*. Some element of T_k contains Y and by condition (1) all of the elements of T_k intersect C_k. Hence C_k intersects R'^* and is therefore a subset of $V^* + A$. With the use of condition (3) it may be seen that $h_1, h_2, h_3, \cdots$ is decreasingly monotonic, hence that each C_{i+1} lies in h_i. But no predecessor of h_i contains A and thus condition (3) shows that h_i is the first region of H_i containing C_{i+1}. By condition (2), each member t of T_k contains an element P of C_k belonging to the boundary of C_k in Σ such that if $i \leqq n$, then t is a subset of every member of T_i that contains P. Clearly, this implies that T_k^* is a subset of V^*. By condition (1), h_k^* is a subset of $T_k^* + C_k$. From these facts it follows that h_k is a subset of R. For some $s > k$ the members of G_s containing Y are subsets of h_k^*. If $i \geqq s$ and the region h of H_i contains A then some element u of $U_{i,h}$ contains Y and is therefore a subset of h_k^*. Since $M_{i,h}$ intersects u it belongs to h_k. So h_k does not precede the first region h' of H_k containing $M_{i,h}$. By condition (3) the region h is a subset of h', which therefore contains A and does not precede h_k. Thus h' is h_k and h is a subset of R. It follows that $H_1, H_2, H_3, \cdots$ is a development of I.

THEOREM 2. *If Σ is complete in sense* C, *so is I.*

PROOF. With $G_1, G_2, G_3, \cdots$ and $H_1, H_2, H_3, \cdots$ as above it follows that if for each n, W_n denotes $H_n + H_{n+1} + H_{n+2} + \cdots$ then $W_1, W_2, W_3, \cdots$ is a decreasingly monotonic development of I. If G' is an infinite subcollection of G and for each n, some element w_n of W_n contains all except finitely many elements of G', then for each n there exists a first region h_n of H_n such that some w_i is a subset of h_n. Let J denote an infinite subset of G'^* such that no member of G' contains two elements of J. In view of condition (1) of the above proof it is clear that for each n the collection U_{n,h_n} is finite, and hence there is a collection T_n of all of its elements having an infinite intersection with J. Moreover, it follows inductively that for each n there is a collection

F_n of all n-term sequences f such that (1) if $i \leqq n$, the ith term of f belongs to T_i, (2) some infinite subset of J intersects each term of f, and (3) if for each $i > 1$ every element of T_i is a subset of some element of T_{i-1} then f is decreasingly monotonic. For each $n > 1$ and sequence f in F_n there exists a sequence f' in F_{n-1} such that if $i \leqq n-1$, the ith term of f' is the ith term of f. Since each collection F_n is finite, there exists a sequence $f_1, f_2, f_3, \cdots$ such that, for each n, f_n belongs to F_n, and if $n > 1$ and $i \leqq n-1$, the ith term of f_{n-1} is the ith term of f_n. There exists a nonrepeating sequence $P_1, P_2, P_3, \cdots$ such that for each n, P_n belongs to J and every term of f_n. It follows inductively that if for each n, R_n denotes the nth term of f_n, then each R_n contains all except finitely many terms of $P_1, P_2, P_3, \cdots$. Each R_n belongs to G_n and thus in view of the conditions on $G_1, G_2, G_3, \cdots$ it may be seen that some element M of G contains a point X such that every domain in Σ to which X belongs contains infinitely many of the points P_n. If the domain R in I contains M then it has an infinite intersection with G', for no set in this collection contains two of the points P_n and R^* is a domain in Σ.

THEOREM 3. *If Σ is complete in sense* A, *so is I.*

PROOF. Retaining the above notation and requiring that $w_1, w_2, w_3, \cdots$ be decreasingly monotonic we see that if $n > 1$ there exists some $i > n$ such that w_i is a subset of h_n and h_{n-1}. For some $k \geqq i$, w_i is an element h of H_k. By condition (3) in the proof of Theorem 1, h is a subset of the first region h' of H_n containing $M_{k,h}$. No term of $w_1, w_2, w_3, \cdots$ is a subset of a predecessor of h_n and the point $M_{k,h}$ lies in h_n. So h' is h_n. Similarly, h_{n-1} is the first element of H_{n-1} containing $M_{k,h}$. Hence, by condition (3), h_n is a subset of h_{n-1} and this, for analogous reasons, implies that h_{n-1} is the first element of H_{n-1} containing M_{n,h_n}. Therefore, by condition (2), each element of T_n is a subset of some element of T_{n-1}. So by definition of the collections F_n each R_{n+1} is a subset of R_n. Thus G contains an element M as in the above proof.

CONSEQUENCES. Terminology appearing below that is not explicitly cited or defined here is much as in [4].

A space satisfies Axiom 1_3 of [6] if and only if it is a regular T_1 space Σ as in the hypothesis of Theorem 1. A space satisfies Axiom 1 if and only if it is a regular T_1 space Σ as in the hypothesis of Theorem 2. In these spaces, a closed point set is compact [1] if and only if it is bicompact [6, Chapter 1]. So if Σ satisfies Axiom 1_3, I is regular and T_1. Moreover, it may be seen that if β is the boundary of an element of an upper semicontinuous decomposition of a regular T_1 space and the decomposition space satisfies the first axiom of countability, then β is bicompact if and only if every collection of domains covering β is refined by a point countable collection of domains covering β. Thus the following theorem is a corollary to Theorems 1 and 2.

COROLLARY 1. *If* (1) *U is an upper semicontinuous decomposition of a space satisfying Axioms* 0 *and* 1_3 (*Axioms* 0 *and* 1) *and* (2) *any collection of domains covering the boundary β of an element of U not a domain is refined by a point countable collection of domains covering β, these conditions are equivalent*:

(1) *The decomposition space satisfies Axiom* 1_3 (*Axiom* 1).

(2) *The decomposition space satisfies the first axiom of countability.*

(3) *No element of U has a noncompact boundary.*

The hypothesis of Corollary 1 involving point countable refinements is nonsuperfluous: There exists an upper semicontinuous decomposition of a certain space satisfying Axioms 0 and 1 such that, while the decomposition space is metrizable and compact, the decomposition contains an element with a noncompact boundary [6, (implicitly) p. 66].

Normalcy is preserved by every upper semicontinuous decomposition of a normal space. Moreover, if β is the boundary of an element of an upper semicontinuous decomposition of a normal T_1 space S and the decomposition space satisfies the first axiom of countability, then β is compact [7]. If, additionally, S satisfies Axiom 1_3, β is bicompact. So the following theorem is a corollary to Theorems 1 and 2.

COROLLARY 2. *If U is an upper semicontinuous decomposition of a normal space satisfying Axioms* 0 *and* 1_3 (*Axioms* 0 *and* 1), *these conditions are equivalent*:

(1) *The decomposition space is a normal space satisfying Axioms* 1_3 (*Axiom* 1).

(2) *The decomposition space satisfies the first axiom of countability.*

(3) *No element of U has a noncompact boundary.*

Collectionwise normalcy [2] is preserved by upper semicontinuous decompositions of collectionwise normal spaces. Every collectionwise normal space satisfying Axioms 0 and 1_3 is metrizable [2]. Every metrizable space complete in sense C is metrically topologically complete [9]. The next theorem follows from these theorems and Corollary 2.

COROLLARY 3 (MORITA-HANAI [7, THEOREM 1] AND A. H. STONE [12, THEOREM 1]; I. A. VAĬNŠTEĬN [13, THEOREM 6]). *If U is an* *upper semicontinuous decomposition of a space that is metrizable (metrically topologically complete), these conditions are equivalent*:

(1) *The decomposition space is metrizable (metrically topologically complete).*

(2) *The decomposition space satisfies the first axiom of countability.*

(3) *No element of U has a noncompact boundary.*

The theorem of Morita-Hanai-Stone may also be derived from Corollary 1, the paracompactness of the closed continuous images of paracompact Hausdorff spaces [5], and the Nagata-Smirnov theorem [8], [11].

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