PROOF. Let X_1 be the set of all x in X for which $|x| = 1$. Since each $G_j \subseteq G = \text{gp } X$, we conclude by Lemma 8 that each $G_j \subseteq \text{gp } X_1$, and hence that $G = \text{gp } X_1$. It follows from the irreducibility of X that $X = X_1$.

This completes the proof of Grushko's Theorem.

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UNIVERSITY OF MICHIGAN

GENERALIZED FUNCTIONS OF SYMMETRIC MATRICES

MARVIN MARCUS[1] AND MORRIS NEWMAN

1. Introduction. In an abstract published in 1961 [4] we announced the following result:

Let A be an n-square positive semi-definite matrix and assume that $A \geqq S$ where S is doubly stochastic. Then

$$\text{per}(A) \geqq n!/n^n. \tag{1.1}$$

The notation $A \geqq S$ means $a_{ij} \geqq s_{ij}$, $i, j = 1, \cdots, n$. A doubly stochastic (d.s.) matrix has non-negative entries and every row and column sum is 1. The permanent, per (A), is the function defined by

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \tag{1.2}$$

where the summation extends over the whole symmetric group of degree n, S_n.

In 1962 [3] we also proved that:

If S is an n-square positive semi-definite symmetric matrix which is doubly stochastic in the extended sense then

$$\text{per}(S) \geqq n!/n^n. \tag{1.3}$$

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A matrix is d.s. in the extended sense if every row and column sum is 1; however, the elements need not all be non-negative. The inequality (1.3) constitutes what is currently known about a conjecture of van der Waerden that states:

$$\operatorname{per}(S) \geqq n!/n^n$$

for any n-square d.s. matrix S.

Observe that for the inequality (1.1) no assumption is made about S being positive semi-definite, otherwise we obviously could get the result from (1.3). One might wonder that if a d.s. matrix S exists for which $A \geqq S$, then perhaps a positive semi-definite matrix S_1 exists which is d.s. in the extended sense and satisfies $A \geqq S_1$. In other words one could hope to relax the d.s. condition in exchange for positive semi-definiteness. This is unfortunately not true. For, take

$$A = \begin{pmatrix} \frac{1}{3} & 0 & \frac{3}{4} \\ 0 & 1 & 0 \\ \frac{3}{4} & 0 & 3 \end{pmatrix}$$

which is obviously positive semi-definite. Clearly $A \geqq S$ where

$$S = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix}.$$

S is not positive semi-definite. Thus we can try for an S_1 such that $A \geqq S_1$, S_1 is d.s. in the extended sense, and S_1 is positive semi-definite. Set

$$S_1 = \begin{pmatrix} a & c & 1-(a+c) \\ c & b & 1-(b+c) \\ 1-(a+c) & 1-(b+c) & a+b+2c-1 \end{pmatrix}.$$

If S_1 is to be positive semi-definite then

$$0 \leqq a(a+b+2c-1) - (1-a-c)^2,$$

which simplifies to

$$(b+1)a \geqq (c-1)^2. \tag{1.4}$$

If $A \geqq S_1$ then $ab+a \leqq \frac{2}{3}$ and $c \leqq 0$, which are incompatible with (1.4).

Since the appearance of the original abstract we have done a substantial amount of work on *generalized matrix functions* as originally defined by I. Schur [7]. Thus let G be a subgroup of S_n and let χ be a character (of arbitrary degree) of G. Following Schur we define the generalized matrix function d_χ by

$$d_\chi(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \tag{1.5}$$

for any n-square matrix A. Clearly if $G = S_n$ and χ is the character of G identically 1 then $d_\chi(A) = \text{per}(A)$. Schur [7, Theorem 1] has shown that $d_\chi(A) > 0$ for positive definite hermitian A. It follows that $d_\chi(A) \geqq 0$ for positive semi-definite A. The purpose of this paper is to prove the following:

THEOREM. *Let A be an n-square positive semi-definite real symmetric matrix and assume that $A \geqq S$ where S is doubly stochastic. Then*

$$d_\chi(A) \geqq m(\chi)/n^n \tag{1.6}$$

where

$$m(\chi) = \sum_{\sigma \in G} \chi(\sigma).$$

For $G = S_n$ and $\chi \equiv 1$, $m = m(\chi) = n!$, and (1.6) specializes to (1.1). It is always true of course that $m(\chi)$ is either 0 or a positive integral multiple of the order of G. More precisely let $\chi = \chi_1 + \cdots + \chi_k$ be a representation of the character χ as a sum of irreducible characters. Then $m(\chi)/g$ is the number of χ_i in this representation which equal the trivial character 1. Here g is the order of G. Thus for nontrivial irreducible χ our theorem reduces to Schur's result.

2. **Preliminary results.** The proof of the inequality (1.6) depends on three theorems of interest in themselves.

THEOREM 1. *If V and W are arbitrary n-square complex matrices then*

$$| d_\chi(VW) |^2 \leqq d_\chi(VV^*) d_\chi(W^*W). \tag{2.1}$$

This result has appeared as a research announcement [2].

THEOREM 2. *If A is a symmetric n-square matrix and $A > 0$ (i.e., $a_{ij} > 0$, $i, j = 1, \cdots, n$) then there exists a unique diagonal matrix*

$$D = \text{diag}(d_1, \cdots, d_n),\ d_i > 0,\ i = 1, \cdots, n, \textit{ such that } DAD \textit{ is d.s.}$$

The authors knew this theorem at the time of the announcement [4] and shortly thereafter a constructive proof was found by Maxfield and

Minc [6]. Independently Sinkhorn [8], [9] proved a closely related result.

THEOREM 3. *Let D be the diagonal matrix described in Theorem 2. Moreover, assume that $A \geqq S$ where S is d.s. Then*

$$\prod_{i=1}^{n} d_i \leqq 1. \tag{2.2}$$

PROOF. For any n-square matrix X let $r_i(X)$ denote the ith row sum of X. Now $A \geqq S$ implies that $DAD \geqq DSD$ and since DAD is d.s. we have

$$1 = r_i(DAD) \geqq r_i(DSD) = d_i \sum_{j=1}^{n} s_{ij} d_j.$$

Thus

$$1 \geqq \prod_{i=1}^{n} d_i \prod_{i=1}^{n} \sum_{j=1}^{n} s_{ij} d_j. \tag{2.3}$$

Let

$$g(X) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} x_{ij} d_j \right)^{1/n}$$

where X is an arbitrary d.s. matrix. It is a well-known result of Birkhoff [1] that the totality Ω_n of n-square d.s. matrices is a convex polyhedron with the permutation matrices as vertices. If X, $Y \in \Omega_n$, $0 \leqq \theta \leqq 1$, then the Hölder inequality implies that

$$\begin{aligned} g(\theta X + (1-\theta)Y) &= \prod_{i=1}^{n} \left(\sum_{j=1}^{n} (\theta x_{ij} + (1-\theta) y_{ij}) d_j \right)^{1/n} \\ &= \prod_{i=1}^{n} \left(\theta \sum_{j=1}^{n} x_{ij} d_j + (1-\theta) \sum_{j=1}^{n} y_{ij} d_j \right)^{1/n} \\ &\geqq \theta g(X) + (1-\theta) g(Y). \end{aligned}$$

Hence g is concave on Ω_n and assumes its minimum on a permutation matrix. The value of g on any permutation matrix is just $\left(\prod_{i=1}^{n} d_i\right)^{1/n}$. From (2.3) we can conclude that

$$1 \geqq \prod_{i=1}^{n} d_i (g(X))^n \geqq \prod_{i=1}^{n} d_i^2$$

and (2.2) follows.

3. **Proof of the Theorem.** In [5] it was proved that if R is any d.s. positive semi-definite matrix then the positive semi-definite determination of the square root of R, $R^{1/2}$, is d.s. in the extended sense. In Theorem 1 let $V=R^{1/2}$ and $W=J_n$, the matrix with every entry $1/n$. Then $R^{1/2}J_n=J_n$, $J_n^2=J_n$, and thus

$$|d_\chi(J_n)|^2 \leqq d_\chi(R)d_\chi(J_n).$$

It is known from Schur's theorems on the d_χ function that $d_\chi(R)\geqq 0$. (R is positive semi-definite.) Hence whether $m(\chi)$ is 0 or not we have

$$d_\chi(R) \geqq d_\chi(J_n) = m(\chi)/n^n. \tag{3.1}$$

We can assume by continuity that $A>0$ in proving (1.6). By Theorem 2 choose a diagonal matrix $D = \operatorname{diag}(d_1, \cdots, d_n)$, $d_i > 0$, $i=1, \cdots, n$ for which $DAD=R$ is d.s. Then

$$d_\chi(DAD) = d_\chi(R) \geqq m(\chi)/n^n. \tag{3.2}$$

But by Theorem 3

$$d_\chi(DAD) = \prod_{i=1}^{n} d_i^2 d_\chi(A) \leqq d_\chi(A). \tag{3.3}$$

The inequalities (3.2) and (3.3) complete the proof.

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University of California, Santa Barbara and
National Bureau of Standards, Washington, D. C.