OPEN SETS OF CONSERVATIVE MATRICES[1]

I. DAVID BERG

In this paper we present two principal results. In Theorem 1 we show that if Γ^l denotes the open set of (conservative) matrices which map some subspace of c of infinite deficiency isomorphically onto c and if Λ denotes the closed set of matrices which sum some bounded divergent sequence, then $[\Gamma^l]^- = \Lambda$. In Theorem 2 we produce a class of triangular matrices with the property that no triangular matrix in a neighborhood of one of these matrices has a range, as an operator on m, whose closure includes c.

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We first introduce some notation, most of which is quite standard.

We denote by m the Banach space of bounded sequences of complex numbers, where $\|x\| = \sup_n |x(n)|$. We denote by c the subspace of m consisting of convergent sequences, by c_0 the subspace of c consisting of sequences with limit 0, and by E^∞ the (nonclosed) subspace of c_0 consisting of sequences with only a finite number of nonzero terms.

We denote by Γ the Banach algebra of conservative matrices (A is called conservative if $x \in c \Rightarrow Ax \in c$), and by Δ the Banach algebra of conservative matrices with zeros above the principal diagonal.

We denote by C_A the vector space of sequences (including unbounded sequences) which are summed by A, that is, transformed into convergent sequences by A. If $C_A \cap m \neq c$ we say $A \in \Lambda$.

When no confusion seems likely to arise we do not differentiate, for $A \in \Gamma$, between A as a transformation from c to c and A as a transformation from m to m. In this regard we note that if $A \in \Gamma$ is considered as an operator from m to m, from c to c, or from c_0 to c, the norm is the same; indeed $\|A\| = \sup_i \sum_j |a_{ij}|$. Furthermore, we recall that, for $A \in \Gamma$, if A^{-1} exists for A as an operator on m, then A^{-1} exists for A as an operator on c and, hence, since A^{-1} is a matrix, $A^{-1} \in \Gamma$ as was shown by A. Wilansky and K. Zeller [3] and by M. R. Parameswaran [5] or as can be seen from our Lemma 1.

We first present the following characterization of Λ.

LEMMA 1. *$A \in \Lambda$ if and only if for $\epsilon > 0$ and integer n there exists $x \in E^\infty$ such that*

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1. $x(1)=x(2)=\cdots=x(n)=0$,
2. $\|x\|=1$,
3. $\|Ax\|<\epsilon$.

PROOF. Let $A\in\Lambda$. Then there exists $x\in m$ such that, if T_nx denotes x with its first n entries replaced by 0,

1. $|\lim_{m\to\infty}(AT_nx)(m)|<1/N<\epsilon/4$,
2. $\|AT_n(x)\|>1$,

for all n.

A sequence of the form

$$\sum_{n=1}^{N}(1/n)T_{m_n}x-\sum_{n=N+1}^{2N}(1/n)T_{m_n}x,$$

for large enough m_n, will satisfy 1, 2, and 3.

The converse is clear by an easy gliding hump construction.

For details the reader is referred to [1]. This completes the proof of Lemma 1.

Lemma 1 yields the result of Wilansky and Zeller and Parameswaran referred to in the introduction when we observe that $A\in\Lambda$ can have no left inverse as an operator from m to m. Hence, if A^{-1} exists, then $A^{-1}(c)=c$.

Lemma 1 also immediately yields the easy result that Λ is closed.

Let $A\in\Gamma$. Suppose there exists some c_1 of infinite deficiency in c such that if the domain of A is cut down to c_1, A is an isomorphism (topological) from c_1 onto c; i.e., if we consider $A: c_1\to c$, A is an isomorphism onto c. We will denote the set of such A by Γ^l.

We will show that the open set Γ^l is dense in Λ.

LEMMA 2. *Γ^l is open.*

PROOF. Γ^l is clearly open, since if $A\in\Gamma^l$, $A: c\to c$ has a right inverse A^r whose range is c_1. Hence, for some neighborhood $A\subset\Gamma$, say V, $B\in V$ implies $BA^r(c)=c$. Hence, $B(c_1)=c$.

LEMMA 3. $\Gamma^l\subset\Lambda$.

PROOF. Let $A\subset\Gamma^l$. Choose an infinite linearly independent set of vectors of the form x_i-y_i where $x_i\in c\sim c_1$ and $y_i\in c_1$ such that $A(x_i-y_i)=0$. It can be easily seen (cf. §3.7 of [4]) that, by a proper choice of scalar λ_i,

$$A\left[\sum_{i=1}^{\infty}\lambda_i(x_i-y_i)\right]=0 \quad\text{and}\quad \sum_{i=1}^{\infty}\lambda_i(x_i-y_i)\in m\sim c.$$

THEOREM 1. $[\Gamma^l]^-=[(\Gamma^l)^0]^-=\Lambda$.

PROOF. By Lemmas 2 and 3, $\Gamma^l = (\Gamma^l)^0$ and $[\Gamma^l]^- \subset \Lambda$. Hence we must show that Γ^l is dense in Λ (which is closed) and our theorem is established.

Let $A \in \Lambda$. By Lemma 1 we may choose a succession of columns of A, $a^{j(0)}, \cdots, a^{j(n)}, \cdots$ as follows: Choose $j(0)$ so that, for some $x_0 \in E^\infty$,

1. $x_0[j(0)] = r_0$, where $|r_0| = 1$,
2. $\|A(x_0)\| < \epsilon/4$.

Choose $j(k)$, $k = 1, 2, \cdots$, so that, for some $x_k \in E^\infty$,

1. $x_k[j(k)] = r_k$, where $|r_k| = 1$,
2. $\|Ax_k\| < (\epsilon/8)(1/2^k)$,
3. $x_k(n) = 0$ if $x_{k-1}(m) \neq 0$ for some $m \geqq n$.

Now define $B \in \Gamma$ as follows: $b^{j(k)} = a^{j(k)} + (\epsilon/4)\delta_k - A(x_k)/r_k$, $k = 0, 1, 2, \cdots$, where δ_0 denotes the constant sequence all of whose entries are 1 and δ_k, $k = 1, 2, \cdots$, denotes the sequence with 1 in the kth place and 0 elsewhere. $b^l = a^l$ for all other l.

It is clear that $B \in \Gamma$ since $B = A + A' + A''$, where $A' \in \Gamma$ is a compact operator and A'' is a submethod of the identity.

$$\|A - B\| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \sum_{k=0}^{\infty} \frac{1}{2^k} = \epsilon.$$

To see that $B \in \Gamma^l$, observe that $B(x_k) = (\epsilon/4)x_k[j(k)]\delta_k = (\epsilon/4)r_k\delta_k$, $k = 0, 1, 2, \cdots$. Hence, if $y \in c$, $\|y\| = 1$,

$$B\left\{(r_0 \lim y)x_0 + \sum_{j=1}^{\infty} (y(j) - \lim y)r_j x_j\right\} = \frac{\epsilon}{4} y.$$

If we denote the above pre-image of y by $B^r y$, we see that $\|B^r y\| < 3 \cdot 4/\epsilon$. Clearly, $B^r(c)$ is isomorphic to c_0. It is also clear that $\{j(k)\}$ can be chosen so as to insure $B^r(c)$ being of infinite deficiency in c.

However, we note that B^r, as defined above, cannot be realized by a matrix.

This completes the proof of Theorem 1.

At this point we note that if we let Γ^{lm} denote the set of $A \in \Gamma$ such that $A : m_1 \to m$ is a (topological) isomorphism onto m from some m_1 of infinite deficiency in m, the arguments of Lemmas 2, 3 and Theorem 1 all go through to show that Γ^{lm} is an open dense set included in Λ. It is rather easy to see that $\Gamma^{lm} \supset \Gamma^l$. To see that the inclusion is proper, consider $A \in \Gamma$ defined by

$$(Ax)(n) = x(2n) - x(2n - 1).$$

It is clear that slight variants of the foregoing arguments lead to other dense open sets in Λ; e.g., the set of A which map m onto a finite deficiency subspace of m, but have infinite dimensional kernel. However, we will not weary the reader with a catalogue of essentially similar results.

We now restrict our attention to Δ.

In Δ, Theorem 1 does not hold; indeed, the set of matrices which are not 1-1 on c is nowhere dense. In Δ, as contrasted with Γ, the maximal group consists of all matrices whose range is all of c.

We are able to present a small class of matrices in $\Lambda\cap\Delta$ which are not approximable in Δ by matrices whose range closure is c, hence which are not on the boundary of the maximal group in Δ. We note that, by Theorem 1, we may approximate such matrices in Γ by matrices whose range is c.

J. Copping on p. 193 of [2] presented an example of an element of $\Delta\cap\Lambda$ which is not on the boundary of the maximal group of Δ.

Copping's example belongs to the class of Nörlund matrices described in the next lemma.

Lemma 4. *Let complex numbers k, l be given, where $|k|>1$, $|l|=1$. Let A be the Nörlund matrix corresponding to $(x-k)(x-l)$. That is:*

$$\begin{aligned} a_{i,i} &= 1, \\ a_{i+1,i} &= -(k+l), \\ a_{i+2,i} &= kl, \\ a_{i,j} &= 0 \quad \textit{for all other } i, j. \end{aligned}$$

Let x_0 be defined by

$$x_0(1) = 1,\ x_0(2) = -l,\ x_0(k) = 0 \textit{ for all other } k.$$

Let $\epsilon>0$ be given. Then there exists $\delta>0$ such that for $B\in\Delta$,

1. $\|B-A\|<\delta$,
2. $\|x-x_0\|<\delta$

jointly imply

$$\left|\frac{y(n+1)}{y(n)} - k\right| < \epsilon,$$

where $y=y(n)$ is defined by $By=x$. Observe that for small enough ϵ, $y\notin m$.

Proof. It suffices to consider the case $l=1$. Consider y defined by $By=x$ where $B\in\Delta$.

Fix n. If we express $y(n+1)$ as $y(n+1)=(1+\eta)ky(n)$ and if we express $y(n+2)$ as $y(n+2)=k(1+\rho)y(n+1)=k^2(1+\rho)(1+\eta)y(n)$, the following equation defines ρ.

$$\sum_{j=1}^{n-1} b_{n+2,j}y(j) + (k + c_1)y(n) - [(k + 1) + c_2]k(1 + \eta)y(n)$$
$$+ k^2(c_3 + 1)(1 + \rho)(1 + \eta)y(n) = x(n + 2),$$

where $c_1=(b_{n+2,n}-a_{n+2,n})$, $c_2=(b_{n+2,n+1}-a_{n+2,n+1})$, $c_3=(b_{n+2,n+2}-a_{n+2,n+2})$.

Solving for ρ we get

$$\rho = \frac{\eta}{k(c_3+1)(1+\eta)} + \frac{ky(n)[(1+\eta)c_2-(k+k\eta)c_3]-\left(\sum_{j=1}^{n-1} b_{n+2,j}y(j)\right)+x(n+2)-c_1y(n)}{k^2(c_3+1)(1+\eta)y(n)}$$

$$= \frac{\eta}{k(c_3+1)(1+\eta)}$$

$+ R$, say (provided all denominators are nonzero).

Choose $\delta_1>0$, $\bar{\eta}>0$ so small that

1. $\bar{\eta}<\min(\epsilon, |k|-1)$,
2. $|c_3|<\delta_1$, $|\eta|<\bar{\eta}$ imply $|\eta/k(1+\eta)(c_3+1)|<2|\eta|/(1+|k|)$.

Now choose δ, $0<\delta<\delta_1$, so small that

1. $\|B-A\|<\delta$, $\|x-x_0\|<\delta$ implies $|y_2/y_1-k|<\bar{\eta}$, $|y_3/y_2-k|<\bar{\eta}$,
2. (i) $|c_i|$, $|x(n+2)|$, $\sum_{j=1}^{n-1}|b_{n+2,j}|<\delta$,
   (ii) $1/2<|y(1)|<\cdots<|y(n)|$,
   (iii) $|\eta|<\bar{\eta}$,

jointly imply

$$R<\frac{1}{2}\left(\frac{\bar{\eta}}{2}-\frac{\bar{\eta}}{1+|k|}\right).$$

Let $\|B-A\|<\delta$, $\|x-x_0\|<\delta$, where δ is chosen as above. (Note that this guarantees that 2(i) holds.)

Suppose $y(j+1)=(1+\eta_j)ky(j)$, where $|\eta_j|<|\bar{\eta}|$, $1\leqq j\leqq n$.

We now show that $y(n+2)=(1+\rho)ky(n+1)$ for some $\rho\in C$ such that $|\rho|<\bar{\eta}$.

If η_n is such that $\bar{\eta}/2<\eta_n<\bar{\eta}$, then $|\rho|<|2\eta/(1+|k|)|+|R|<|\eta|$. If η_n is such that $\eta_n\leqq\bar{\eta}/2$, then $|\rho|<\bar{\eta}/2+|R|<\bar{\eta}/2+\bar{\eta}/2=\bar{\eta}$. This completes the proof of Lemma 4.

THEOREM 2. *Let $A=BGD$ where B, G, $D\in\Delta$ and*

1. *B and D are invertible,*
2. *G is the Nörlund matrix corresponding to $(x-k)(x-l)$, where $|k|>1$.*

Then there is a neighborhood, V, of A in Δ with the following property:

For each $H\in V$, there exists some $f\in l_1$ such that $H^*f=0$, i.e., $fH=0$, where f is written as a row matrix. Equivalently, for $H\in V$, $[H(m)]^-\not\supset c_0$.

PROOF. We first observe that it suffices to prove the theorem for neighborhoods of G. For, if V is a small enough neighborhood of A, $V=BV'D$ where V' is a neighborhood of G. But if $fH=0$ for some $H\in V'$, $fB^{-1}\in l_1$ and $(fB^{-1})F=0$ for $F=BHD$.

We now consider three cases. Cases 1 and 2 are well known.

Case 1. $|l|<1$. This case is clear since G is now an isomorphism of c onto $f^\perp$ where $f=(1, k^{-1}, k^{-2}, \cdots)\in l_1=c_0^*$. Hence there exists $V\subset\Delta$, indeed, $V\subset\Gamma$, such that, for $H\in V$, there exists $g\in l_1$ such that $H^*g=0$.

Case 2. $|l|>1$. This case is, similarly, clear since G is now an isomorphism of c onto $f^\perp\cap h^\perp$ for appropriate $h\in l_1$.

Case 3. $|l|=1$. In this case $G(c)$ is not closed in c so the above arguments do not apply. Indeed, $G(c)$ is dense not closed in $f^\perp$ where $f=(1, k^{-1}, k^{-2}, \cdots)$. But, by Lemma 4, there exists a neighborhood of G, $V\subset\Delta$, such that for $H\in V$ $[H(m)]^-\not\supset x_0$ where x_0 is as defined in Lemma 4. Hence there exists $f\in l_1$ such that $H^*f=0$. This completes the proof of Theorem 2.

In cases 1 and 2 of Theorem 2 the conclusion holds even if neighborhoods were taken in Γ. It is also clear that the arguments in these cases, which are quite standard, did not depend on the degree of the polynomial.

However, Theorem 1 tells us that in case 3 Theorem 2 is false if neighborhoods are taken in Γ. While it seems likely that case 3 is independent of the degree of the polynomial, we have not been able to prove the equivalent theorem for higher degree polynomials.

Theorems 1 and 2 suggest the following question which we cannot answer.

If $A\in\Lambda$ and A is 1-1 on c, is A on the boundary of the maximal group of Γ?

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YALE UNIVERSITY