ON THE TOPOLOGY OF EIII AND EIV

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Introduction. In [4] we introduced the concept of the root diagram of a compact symmetric triad. Now we consider the symmetric triad $(E_6;\ F_4,\ D_5\times T^1)$ (where we represent compact Lie groups by the standard symbols for their local structures and take E_6 to be simply connected). The corresponding compact symmetric spaces, in E. Cartan's notation, are EIII and EIV. We shall determine the root diagram of the triad and then apply Morse theory to obtain topological information. The natural action of F_4 on EIII will be found to have as one of its orbits the Cayley projective plane $W=F_4/B_4$, and the action of $D_5\times T^1$ on EIV will have an orbit $S^1\times S^9$. An analysis of certain spaces of paths will then establish the following theorems and their corollaries.

THEOREM A. $\pi_j(\mathrm{EIII},W) = \pi_{j-1}(S^1) + \pi_{j-1}(S^{16})$, $0<j<32$, *and* $\pi_j(E_6/D_5,\ W)=\pi_{j-1}(S^{16})$, $0<j<32$.

THEOREM B. $\pi_j(\mathrm{EIV},\ S^1\times S^9)=\pi_{j-1}(S^1) + \pi_{j-1}(S^{16})$, $0<j<32$, *and* $\pi_j(\mathrm{EIV},\ S^9)=\pi_{j-1}(S^{16})$, $0<j<32$.

COROLLARY 1. *The natural inclusions* $W\subset\mathrm{EIII}$ *and* $W\subset E_6/D_5$ *induce isomorphisms* $\pi_j(\mathrm{EIII}) = \pi_j(W) = \pi_{j-1}(S^7)$, $2<j<16$, *and* $\pi_j(E_6/D_5)=\pi_j(W)$, $0<j<16$.

(There is a corresponding corollary to Theorem B. Most of the information that it contains also follows from Araki's computation of $H_*(\mathrm{EIV})$ [1]).

COROLLARY 2. $\pi_{16}(\mathrm{EIII})=(\boldsymbol{Z}_2)^k$, $k=2$ or 3; $\pi_{17}(\mathrm{EIII})=(\boldsymbol{Z}_2)^k+\boldsymbol{Z}$, $k=3$ or 4; $\pi_{21}(\mathrm{EIII})=\boldsymbol{Z}_6$.

COROLLARY 3. $\pi_{17}(\mathrm{EIV})=\boldsymbol{Z}+(\boldsymbol{Z}_2)^k$, $k=2$ *or* 3; $\pi_{21}(\mathrm{EIV})=0$; $\pi_{22}(\mathrm{EIV})=\boldsymbol{Z}_6$ *or* $\boldsymbol{Z}_3$.

COROLLARY 4. $H^*(E_6/D_5)=\boldsymbol{Z}[x_8,\ x_{17}]/\{x_8^3,\ x_{17}^2\}$, $\dim(x_j)=j$.

COROLLARY 5.

$$H_j(\mathrm{EIII})=\begin{cases}0, & j\ \text{odd}\\ \boldsymbol{Z}, & j=0,\ 2,\ 4,\ 6,\ 26,\ 28,\ 30,\ 32\\ \boldsymbol{Z}+\boldsymbol{Z}, & j=8,\ 10,\ 12,\ 14,\ 18,\ 20,\ 22,\ 24\\ \boldsymbol{Z}+\boldsymbol{Z}+\boldsymbol{Z} & j=16.\end{cases}$$

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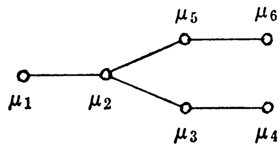
Other homotopy information can be deduced from Theorems A and B. For instance, most of the groups $\pi_j(\mathrm{EIV})$, $j\leqq 27$, can be computed modulo 2-primary components.

In what follows we regularly employ capital Italic letters for Lie groups and the corresponding lower case German letters for their Lie algebras. An invariant positive definite inner product on $\mathfrak{e}_6$ is chosen once and for all, and any subspace of $\mathfrak{e}_6$ is identified with its dual via this inner product.

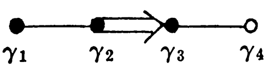
1. **The roots of** $(E_6;\ F_4,\ D_5\times T^1)$. We will sketch very briefly the computation of this root system. Let $\mathfrak{F}$ be a fundamental system of roots for E_6 relative to a maximal torus T, and label $\mathfrak{F}$ as in (1.1). Define involutions J and L

(1.1)



of the root system $R(\mathfrak{F})$ of E_6 as follows. Let L arise from the involution of E_6 whose fixed point group is F_4, so defined that $\mathfrak{F}$ is a $(-L)$-fundamental system (cf. [2]). Let A be the nontrivial involution of $\mathfrak{F}$ (hence an involution of $R(\mathfrak{F})$) and set $J=AL$. One proves (using the theory of [2]) that J comes from an involution of E_6 which defines the symmetric space EIII. We denote these involutions of E_6 again by J and L and continue to denote the product JL by A.

Let $\mathfrak{t}_A$ be the subspace of $\mathfrak{t}$ on which A is the identity. J and L restrict to the same involution $-\sigma$ of $\mathfrak{t}_A$ and the restricted root system in $\mathfrak{t}_A$ is a normal σ-system with Satake figure (cf. [2] for the definitions) as in (1.2).



(1.2) $\quad \gamma_i = \mu_i \big| \mathfrak{t}_A, \qquad i = 1, \cdots, 4; \qquad \gamma_i = \mu_{i+2}\big| \mathfrak{t}_A, \quad i = 3, 4.$

Let $\mathfrak{s}$ be the negative eigenspace of $-\sigma$ in $\mathfrak{t}_A$.

Now the Siebenthal functionals on $\mathfrak{t}_A$ (cf. [4], [6]) with linear parts γ_i can be shown to be γ_1, γ_2, γ_3, $\gamma_3+1/2$, $\gamma_4+\delta$, $\gamma_4+\delta+1/2$ for some non-negative $\delta<1$. As in [4], we can replace J by $\mathrm{Ad}(x)(J)\mathrm{Ad}(x^{-1})$ for suitable $x\in S=\exp(\mathfrak{s})$ and bring about that $\delta=0$. Under this assumption it follows that all of the Siebenthal functionals in $\mathfrak{t}_A$ have constant terms 0 or 1/2.

Setting $\gamma=\gamma_4\big|\mathfrak{s}$, we find that the root system of the triad consists of the functionals γ, $\gamma+1/2$, 2γ, $2\gamma+1/2$ with respective multiplicities

8, 8, 7, 1. The torus in EIII fundamental to the action of F_4 (and likewise that in EIV fundamental to the action of $D_5\times T^1$) is a circle S with four singular points equally spaced about it with respective multiplicities 15, 1, 15, 1. Denote by p and q the two singular points of multiplicity 15.

2. **The orbits of** p. The singular point p can be interpreted as a point of EIII as well as a point of EIV. In the first case we wish to determine the F_4-orbit of p and in the second the $D_5\times T^1$-orbit of p. Note that the automorphism A of E_6 is involutive (since the Siebenthal functionals all have constant terms 0 or 1/2) and consequently J and L commute. Therefore $F_4\cap(D_5\times T^1)$ is a symmetric subgroup of F_4 and of $D_5\times T^1$. Thus the F_4-orbit of p is a symmetric space of F_4 and the $D_5\times T^1$-orbit of p is a symmetric space of $D_5\times T^1$.

Now by the fact that $\dim(\mathrm{EIII})=32$, the dimension of the F_4-orbit of p must be 16. From the classification of symmetric spaces we obtain

(2.1) LEMMA. *$F_4\cap(D_5\times T^1)=B_4$ and the F_4-orbit of p is $W=F_4/B_4$.*

It also follows that the $D_5\times T^1$-orbit is an S^9-bundle over S^1. It is not hard to show that it is actually a trivial bundle.

(2.2) LEMMA. *The $D_5\times T^1$-orbit of p is $S^1\times S^9$.*

(2.3) LEMMA. *p and q lie on the same K-orbit, $K=F_4$ or $D_5\times T^1$.*

PROOF. By a basic transformation as defined in [**4**], a singular point of multiplicity one in $\mathfrak{S}$ can be translated to the origin. Then there is an action of K which stabilizes the point of S corresponding to the origin of $\mathfrak{S}$ and reflects S through that point (just consider the Weyl group for the symmetric space defined by restricting J to the fixed point group of A). This action, under the inverse of our basic transformation, becomes an action of K carrying p to q. q.e.d.

3. **Proofs of Theorems A and B.** Consider the space of paths $\Omega(\mathrm{EIII}; x, W)$ (defined as in [**3**]) where x is a nonsingular point of S. Then by §1 and (2.3) the (increasing) sequence of integers which occur as Morse indices for transversal geodesic segments in Ω begins with 0, 1, 16, 17, 32, $\cdots$. Thus the Morse theory gives

$$\pi_j(\mathrm{EIII},W) = \pi_{j-1}(S^1 \vee S^{16} \cup_f e_{17}), j < 32, \tag{3.1}$$

where $\vee$ denotes the one-point union and f is an attaching map taking the boundary of the 17-cell into $S^1\vee S^{16}$. Theorem A will follow from (3.1) and the following proposition.

(3.2) PROPOSITION. *There is a homotopy equivalence $S^1\vee S^{16}\cup_f e_{17} \simeq S^1\times S^{16}$.*

As is well known, there exists an attaching map g such that $S^1 \vee S^{16} \cup_g e_{17} = S^1 \times S^{16}$, so we have to show that f is homotopic to g. We proceed by a series of lemmas.

Consider the fibration $\pi: E_6/D_5 \rightarrow \mathrm{EIII}$, the fiber being S^1. The natural injection $F_4/B_4 \rightarrow E_6/D_5$ followed by π gives our imbedding of W in EIII. The following lemma is established by these facts and the five lemma.

(3.3) LEMMA. π: $(E_6/D_5, W) \rightarrow (\mathrm{EIII}, W)$ *induces bijections in* π_j, $j > 2$.

(3.4) LEMMA. $\pi_j(E_6/D_5, W) = 0$, $0 < j < 17$.

PROOF. For $2 < j < 17$ this follows from (3.1) and (3.3). The equalities $\pi_1(E_6/D_5) = 0$, $\pi_0(W) = 0$ give $\pi_1(E_6/D_5, W) = 0$. Similarly, $\pi_2(E_6/D_5) = 0$ and $\pi_1(W) = 0$ imply that $\pi_2(E_6/D_5, W) = 0$. q.e.d.

(3.5) COROLLARY. $\pi_{17}(E_6/D_5, W) = H_{17}(E_6/D_5, W)$.

PROOF. Both spaces are simply connected, so the result follows from (3.4) by the relative Hurewicz theorem. q.e.d.

(3.6) LEMMA. $H_{17}(E_6/D_5, W) = H_{17}(E_6/D_5)$.

PROOF. Consider the following commutative diagram where h_1 and h_2 are Hurewicz maps:

$$\begin{array}{ccc} \pi_{17}(E_6/D_5, W) & \xrightarrow{\delta_2} & \pi_{16}(W) \\ \downarrow h_1 & & \downarrow h_2 \\ H_{17}(E_6/D_5, W) & \xrightarrow[\delta_1]{} & H_{16}(W) \end{array}$$

Since $\pi_{16}(W) = \pi_{15}(\Omega(W)) = \pi_{15}(S^7)$ (using the root diagram of W as given in [2] together with Morse theory) and this group is $(\boldsymbol{Z}_2)^3$(cf. [7]), and since $H_{16}(W) = \boldsymbol{Z}$, it follows that $h_2 = 0$. h_1 being bijective (3.5), it follows that $\delta_1 = 0$. Finally, the equality $H_{17}(W) = 0$ together with the exact sequence in homology gives the lemma. q.e.d.

(3.7) LEMMA. $H_{17}(E_6/D_5) = \boldsymbol{Z}$.

PROOF. Using $\delta_1 = 0$ from the proof of (3.6) and the fact that $H_{16}(E_6/D_5, W) = 0$ ((3.4) and Hurewicz) we obtain that $H_{16}(E_6/D_5) = H_{16}(W) = \boldsymbol{Z}$. Similarly $H_{15}(E_6/D_5) = 0$. Poincaré duality gives $H^{18}(E_6/D_5) = 0$, $H^{17}(E_6/D_5) = \boldsymbol{Z}$. Since E_6/D_5 is a finite complex, the universal coefficient theorem dualizes to give

$$H_{17}(E_6/D_5) = \mathrm{Hom}(H^{17}(E_6/D_5), \boldsymbol{Z}) + \mathrm{Ext}(H^{18}(E_6/D_5), \boldsymbol{Z}) = \boldsymbol{Z}.$$

q.e.d.

(3.8) COROLLARY. $\pi_{16}(\Omega(\mathrm{EIII}; x, W)) = \boldsymbol{Z}$.

This last corollary contains the essential information about the attaching map f. By [5, p. 145] there is a canonically split exact sequence

$$0 \to \pi_{17}(S^1 \times S^{16}, S^1 \vee S^{16}) \xrightarrow{\delta} \pi_{16}(S^1 \vee S^{16}) \underset{j}{\rightleftarrows} \pi_{16}(S^1 \times S^{16}) \to 0.$$

Let α generate $\pi_{17}(S^1 \times S^{16}, S^1 \vee S^{16}) = \boldsymbol{Z}$ and let β generate $\pi_{16}(S^1 \times S^{16}) = \boldsymbol{Z}$. Set $\alpha' = \delta(\alpha)$, $\beta' = j(\beta)$, and note that α' is just $\pm$ the homotopy class of the attaching map g already alluded to. Let

$$r: S^1 \vee S^{16} \to (S^1 \vee S^{16}) \cup_f e_{17}$$

be the inclusion. The map f defines an element $[f] \in \pi_{16}(S^1 \vee S^{16})$ which can be written $m\alpha' + n\beta'$ for suitable integers m and n. Therefore

$$(3.9) \qquad mr_*(\alpha') + nr_*(\beta') = 0.$$

Now remark that the triad $(E_6; F_4, D_5 \times T^1)$ is regular in the sense of [4]. Thus by Theorem 2.2 of [4], the homology results of Bott and Samelson [3] hold for $\Omega(\mathrm{EIII}; x, W)$ over the integers. In particular, if h denotes the Hurewicz map, the spherical homology class $h(r_*(\beta'))$ is the generator of $H_{16}(S^1 \vee S^{16} \cup_f e_{17}) = \boldsymbol{Z}$. It is easy to see that $h(r_*(\alpha')) = 0$, so (3.9) implies that $n = 0$.

(3.10) LEMMA. $[f] = m\alpha'$.

(3.11) COROLLARY. $[f] = \pm\alpha'$.

PROOF. By (3.10), $\pi_{16}(S^1 \vee S^{16} \cup_f e_{17}) = \boldsymbol{Z}_m + \boldsymbol{Z}$. By (3.8), $m = \pm 1$. q.e.d.

Recalling that α generates $\pi_{17}(S^1 \times S^{16}, S^1 \vee S^{16})$, we can conclude to (3.2) by standard theory. This completes the proof of Theorem A.

The proof of Theorem B is quite similar, but a little easier. One verifies that the inclusion $(\mathrm{EIV}, S^9) \subset (\mathrm{EIV}, S^1 \times S^9)$ induces bijections in π_j, $j > 2$, and that $\pi_j(\mathrm{EIV}, S^9) = 0$, $0 < j < 17$. It follows that $\pi_{17}(\mathrm{EIV}, S^9) = H_{17}(\mathrm{EIV}, S^9)$.

(3.12) LEMMA. $H_{17}(\mathrm{EIV}, S^9) = \boldsymbol{Z}$.

PROOF. We have $H_{17}(S^9) = H_{16}(S^9) = 0$, and so $H_{17}(\mathrm{EIV}, S^9) = H_{17}(\mathrm{EIV})$. This last group is $\boldsymbol{Z}$ by [1, Lemma 2.4]. q.e.d.

(3.13) COROLLARY. $\pi_{16}(\Omega(\mathrm{EIV}; x, S^1\times S^9))=\boldsymbol{Z}$.

This corollary contains all the necessary information about the pertinent attaching map exactly as in the proof of Theorem A, so we obtain Theorem B.

4. **Proofs of the corollaries.** Corollary 1 is immediate from Theorem A and the equality $\pi_j(W)=\pi_{j-1}(S^7)$, $j<22$, which follows by Morse theory from the root diagram of W.

Corollaries 2 and 3 are proven from the exact homotopy sequence of a pair together with Toda's computations of homotopy of spheres [7].

Now Corollary 1, together with the fact $\delta_1=0$ proven in (3.6), shows that the inclusion $W\subset E_6/D_5$ induces isomorphisms $H_j(W)=H_j(E_6/D_5)$, $0\leqq j<17$. Poincaré duality and the universal coefficient theorem then yield the group-theoretic part of Corollary 4. Let x_8 generate $H^8(E_6/D_5)=\boldsymbol{Z}$. The fact that x_8^2 generates $H^{16}(E_6/D_5)=\boldsymbol{Z}$ then follows from the ring structure of $H^*(W)$ and the fact that the isomorphism $H^j(W)=H^j(E_6/D_5)$, $0\leqq j<17$, is induced by the inclusion. If x_{17} generates $H^{17}(E_6/D_5)=\boldsymbol{Z}$, then Poincaré duality proves that $x_8^2x_{17}$ generates $H^{33}(E_6/D_5)=\boldsymbol{Z}$. The following lemma completes the proof of Corollary 4.

(4.1) LEMMA. *x_8x_{17} generates $H^{25}(E_6/D_5)$.*

PROOF. Let $\mu\in H_{33}(E_6/D_5)$ be the fundamental class. Then by Poincaré duality $y=x_8\cap\mu$ generates $H_{25}(E_6/D_5)=\boldsymbol{Z}$. By the natural pairing of homology and cohomology,

$$1=\langle x_8^2x_{17},\mu\rangle=\langle x_8x_{17}, x_8\cap\mu\rangle=\langle x_8x_{17}, y\rangle.$$

q.e.d.

Recalling that $\pi\colon E_6/D_5\to\mathrm{EIII}$ is a circle bundle, we can deduce Corollary 5 in dimensions <17 from Corollary 4 by a standard (and easy) spectral sequence argument. Poincaré duality then completes the proof of Corollary 5.

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A NOTE ON COMPACT TRANSFORMATION GROUPS WITH A FIXED END POINT

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Dedicated to A. D. Wallace for his 60th birthday

1. Introduction. In [**2**], Professor A. D. Wallace proved the following: "Let T be a cyclic transformation group of a Peano continuum X leaving fixed an end point, then T has another fixed point." In [**4**], Professor H. C. Wang arrived at the same result by assuming that T is compact and X is an arcwise connected Hausdorff space. In this note, under the same assumption as Wang's, we prove that T has countably many fixed points. In fact, we prove the following

Theorem. *Let (X, T, π) be a transformation group where X is an arcwise connected Hausdorff space. Let A be a closed T-invariant set which is separated from any other closed T-invariant set B, $B \cap A = \varnothing$, by a point. If there is such a closed set B, then T has at least two distinct fixed points, one of them contained in A. If, furthermore, every orbit, under T, is closed, then T has countably many fixed points.*

2. Proof of the theorem. The main technique of the proof is based on the proof used in [**4**] with some modification. Choose $a \in A$ and $b \in B$; connect a and b by an arc $l(t)$, $0 \leqq t \leqq 1$, with $l(0) = a$ and $l(1) = b$. Let S be the set of all points which separate A and B. Then, by our assumption, S is not empty. It is clear that S lies on the arc $l(t)$, as does $\text{cl}(S)$. It is also obvious that $g(S) = S$ and $g(\text{cl}(S)) = \text{cl}(S)$ for

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