SHORTER NOTES

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A PROOF OF THE MARTINGALE CONVERGENCE THEOREM[1]

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In this note we give another proof of the martingale convergence theorem. The proof does not use the usual "upcrossing" lemma but does make use of the classical martingale inequalities (3) and (4). We employ the following theorem, essentially proved in [2] (Satz 3.3, p. 131).

THEOREM 1. *Let* $\{X_n\}$ *be a martingale. Then there is a representation* $X_n = U_n - V_n$ *where* U_n *and* V_n *are non-negative martingales if and only if* $\lim E|X_n| < \infty$.

The scheme of the proof is as follows: first we show that a non-negative submartingale with bounded second moments is a mean square Cauchy sequence, next that such a submartingale converges a.s. We then prove convergence for non-negative martingales and finally Theorem 1 proves the general case. The basic reference is [1] except that we use the term "submartingale" instead of "semi-martingale." Only real-valued random variables will be considered throughout.

LEMMA 1. *Let* $\{X_n\}$ *be a non-negative submartingale with* $\lim EX_n^2 < \infty$. *Then* $\lim_{n,m} E|X_n - X_m|^2 = 0$.

PROOF. First remark that X_n^2 is a submartingale, so that EX_n^2 is monotone nondecreasing. Letting $n > m$, we have

$$\begin{aligned} EX_n^2 - EX_m^2 &= E\{X_m + (X_n - X_m)\}^2 - EX_m^2 \\ &= E\{X_m^2 + 2X_m(X_n - X_m) + (X_n - X_m)^2\} - EX_m^2 \qquad (1) \\ &= 2E\{X_m(X_n - X_m)\} + E|X_n - X_m|^2. \end{aligned}$$

Write $X_j = \sum_{i=1}^{j} Y_i$ where $E\{Y_{m+1} + \cdots + Y_n | Y_1, \cdots, Y_m\} \geqq 0$ for every m and $n > m$. Then the first term on the right of (1) can be written

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\begin{aligned}
E\{X_m(X_n - X_m)\} &= E\{E\{X_m(X_n - X_m) \mid Y_1, \cdots, Y_m\}\} \\
&= E\{X_m E\{Y_{m+1} + \cdots + Y_n \mid Y_1, \cdots, Y_m\}\} \geqq 0.
\end{aligned} \tag{2}
$$

The last inequality follows since the random variable $X_m E\{Y_{m+1} + \cdots + Y_n \mid Y_1, \cdots, Y_m\}$ is non-negative. As m and n tend to ∞, the left side of (1) converges to zero and, because of (2), the desired conclusion follows.

LEMMA 2. *Let* $\{X_n\}$ *be a non-negative submartingale with* $\lim EX_n^2 < \infty$. *Then* $\lim X_n = X$ *a.s.*

PROOF. If $\{Z_j, 1 \leqq j \leqq n\}$ is a submartingale and ϵ is any real number, we have [1, p. 314]

$$\epsilon P\{\max_j Z_j \geqq \epsilon\} \leqq E \mid Z_n \mid, \tag{3}$$

$$\epsilon P\{\min_j Z_j \leqq \epsilon\} \geqq EZ_1 - E \mid Z_n \mid. \tag{4}$$

If $n > m$, then $\{X_j - X_m, m < j \leqq n\}$ is a submartingale. Using (3) and (4) we obtain

$$
\begin{aligned}
P\{\max_{m<j\leqq n} \mid X_j - X_m \mid \geqq \epsilon\} &\leqq P\{\max_{m<j\leqq n} \{X_j - X_m\} \geqq \epsilon\} \\
&\quad + P\{\min_{m<j\leqq n} \{X_j - X_m\} \leqq -\epsilon\} \\
&\leqq \frac{2E \mid X_n - X_m \mid - E\{X_{m+1} - X_m\}}{\epsilon} \\
&\leqq \frac{2E \mid X_n - X_m \mid + \mid E\{X_{m+1} - X_m\} \mid}{\epsilon}.
\end{aligned}
$$

$E \mid X_n - X_m \mid \leqq \{E \mid X_n - X_m \mid^2\}^{1/2} \to 0$ as m and n tend to ∞, by Lemma 1. Also, $\mid E\{X_{m+1} - X_m\} \mid \leqq E \mid X_{m+1} - X_m \mid \to 0$ as m tends to ∞. The proof may now be completed along the lines of Theorem 2.3, p. 108 of [1].

LEMMA 3. *Let* $\{X_n\}$ *be a non-negative martingale. Then* $\lim X_n = X$ *a.s.*

PROOF. $Y_n = \exp(-X_n)$ is a submartingale, $0 \leqq Y_n \leqq 1$. The hypotheses for Lemma 2 are satisfied for $\{Y_n\}$, and so $\lim Y_n = Y$ a.s. If $Y = 0$ on a set of positive probability, then $X_n \to \infty$ on this set, which implies $\lim EX_n = \infty$, contradicting the definition of a martingale. Hence $Y > 0$ a.s., and we have, using the continuity of the log function

$$\lim X_n = \lim (-\log Y_n) = -\log Y \text{ a.s.}$$

THEOREM 2 (DOOB). *Let* $\{X_n\}$ *be a martingale,* $\lim E|X_n| < \infty$. *Then* $\lim X_n = X$ *a.s.*

PROOF. Theorem 1 and Lemma 3 prove the theorem.

REFERENCES

1. J. L. Doob, *Stochastic processes*, Wiley, New York, 1953.
2. K. Krickeberg, *Wahrscheinlechkeits-theorie*, Teubner, Stuttgart, 1963.

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