CONCERNING THE COMMUTATOR SUBGROUP OF A RING

W. E. BAXTER

This paper considers two independent results concerning $[A, A]$, the commutator subgroup of an associative ring A, and generated by all elements $[a, b]=ab-ba$, where a and b are in A. The first of these results sharpens those of [**3**], while the second uses the techniques of [**6**] to generalize [**1**] and [**4**]. These results are stated as

THEOREM 1. *Let A be a simple associative ring; then either A is a field or $[A, A]^2$, the subgroup generated by all products ab where a and b are in $[A, A]$, is A.*

THEOREM 2. *Let A be an associative ring such that $[A, A]^-$, the subring generated by $[A, A]$, is A and let U be a Lie ideal of $[A, A]$, then either $[[U, U], U]=(0)$ or there exists a nontrivial (two-sided) ideal, R, of A such that $R\subset U^-$.*

PROOF OF THEOREM 1. Assume A is not 4-dimensional over Z, its center and a field of characteristic 2; if so, then a direct verification shows that $[A, A]^2=A$. Let $x, y\in[A, A]$ and $a\in A$, then $[x, y]a = [x, ya]+y[a, x]$. Thus,

$$(xy - yx)A \subset [A, A] + [A, A]^2 \quad \text{for all } x, y \in [A, A].$$

Now for any $b\in A$, $b[x, y]a = [b, [x, y]a]+[x, y]ab$ and hence,

$$A(xy - yx)A \subset [A, A] + [A, A]^2 \quad \text{for all } x, y \in [A, A].$$

Therefore either (a) $[[A, A], [A, A]]=(0)$, or (b) $A=[A, A] +[A, A]^2$. (a) implies by [**4**] and [**1**] that A is a field, and (b) implies $[A, A]^2=A$ by the use of the following lemma.

LEMMA 1 (HERSTEIN). *Let A be a simple associative ring, neither a field nor 4-dimensional over its center, Z, a field of characteristic 2. Then $[A, A]\subset[A, A]^2$.*

PROOF. $[A, A]^2$ is obviously a Lie ideal of A and hence by [**3**] either is contained in Z or contains $[A, A]$. We now show that $[A, A]^2\subset Z$ leads to a contradiction. Let $a, b, c\in A$; then $u=[a, b][a, c]$ and $ua=[a, b][a, ca]$ are in Z. Now if $u\neq 0$, then the latter implies that $a\in Z$ and hence $u=0$, which is false. Thus, for all a, b, c in A,

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$[a, b][a, c]=0$. An easy verification shows that this leads to $a\in Z$, a contradiction. Thus the desired conclusion.

PROOF OF THEOREM 2. We assume that $[A, A]^-=A$. To prove the theorem we need the following lemma.

LEMMA 2. *Let U be a Lie ideal of $[A, A]$. Then $I = I(U) = \{u\in U^- \mid ua\in U^- \text{ for all } a\in A\}$ is an ideal of A with the property that it contains every ideal of A which is a subset of U^-.*

PROOF. The latter statement is obvious from the definition of I. It is also evident that I is a right ideal. Let $b\in[A, A]$, $a\in A$, and $u\in I$. Then, $b(ua)-(ua)b\in U^-$, and $bu-ub\in U^-$ which implies that $[A, A]I\subseteq I$. Thus, for all $n\geqq 1$, $[A, A]^n[A, A]^nI\subseteq I$ and hence $AI\subseteq I$. So, I is an ideal of A. (The lemma also holds with U replacing U^- everywhere in the definition of I.)

We are now in a position to prove Theorem 1. Suppose $[[U, U], U]\neq(0)$; then there exists $x\in[U, U]$, $y\in U$ such that $xy-yx\neq 0$. Since $[[U, U], A]\subseteq[U, [U, A]]\subseteq U$, we have $[x, y]\in U$. Also, $[x, y]a=[x, ya]+y[x, a]$ for all $a\in A$. By the previous remark, $[x, ya]$ and $[x, a]$ are in U and thus $[x, y]a\in U^-$ for all $a\in A$. Thus, $I\neq(0)$, and by Lemma 2, the theorem is proved.

This theorem can be strengthened to Theorem 3 for certain rings using an argument similar to [**3**] and the following lemma.

LEMMA 3 [**5**]. *If a ring A has no nonzero right ideal, J, with $a^n=0$ for all $a\in J$, n fixed, then A has a nonzero nilpotent (two-sided) ideal.*

THEOREM 3. *Let A be a ring with no nilpotent ideals and such that $2x=0$ implies $x=0$. Then either U^- contains a nontrivial ideal of A or $[U, U]\subset Z$, the center of A.*

PROOF. We have seen that $[x, y]\in I$ for all $x\in[U, U]$, $y\in U$. Thus, either U^- contains a nontrivial ideal of A or $[x, y]=0$ for all $x\in[U, U]$, $y\in U$. If the latter holds, then for all $a\in A$, $[x, [x, a]]=0$. Setting $a=bc$ and expanding the resulting expression, we obtain $2[x, b][x, c]=0$ for all $b, c\in A$ which yields, using the hypothesis,

$$[x, b]^2 = 0 \quad \text{for all } x\in[U, U],\ b\in A. \tag{1}$$

Suppose $[x, a]=0$, $x\in[U, U]$, and for all $a\in[A, A]$; then, since $[A, A]^-=A$, $x\in Z$. Thus, assume that $y=[x, b]\neq 0$ for some $b\in[A, A]$. Then, $y\in[U, U]$ and from (1) we have

$$y^2 = 0 \quad \text{and} \quad [y, d]^2 = 0 \quad \text{for all } d\in A. \tag{2}$$

Multiply (2) on the left by y and on the right by d and obtain $(yd)^3=0$. Thus yA is a right ideal satisfying identity of Lemma 3. If $yA\neq(0)$, then we have a contradiction, while $yA=(0)$ implies (A being simple) that $y=0$, which also is a contradiction. Thus we have shown $[U, U]\subset Z$.

This result indeed generalizes the work of [**1**] and [**4**].

THEOREM 4. *If A is simple (then $[A, A]^-=A$) and U is a proper Lie ideal of $[A, A]$, then U is contained in the center of A except where A is of characteristic 2 and 4-dimensional over Z, a field of characteristic 2.*

PROOF. Define $[U, U]=U^{(1)}$ and $U^{(n+1)}=[U^{(n)}, U^{(n)}]$ for all $n\geqq 1$. Then, since A is simple, it has no nonzero nilpotent ideals. Thus, except in characteristic 2, $[U, U]\subset Z$ or $U^-=A$. If the former, then Theorems 7 and 9 of [**4**], in the case not characteristic 3, and Lemma 3 of [**1**] in this case implies $U\subset Z$. Now, by these same results, if $U^{(2)}\subset Z$, then $U\subset Z$. Hence $\{U^{(2)}\}^-=A$. Thus, by Lemma 9 of [**2**] we have $[U^{(2)}, A]=[A, A]$, which contradicts U being proper. Lemma 1 of [**1**] yields the result when A is of characteristic 2.

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UNIVERSITY OF DELAWARE