FACTOR SETS FOR DOUBLY STOCHASTIC OPERATORS ON A HILBERT SPACE

RICHARD SINKHORN

A doubly stochastic matrix is usually defined as an $n \times n$ matrix which has nonnegative elements and row and column sums one. If the restriction on the nonnegativity of the elements is ignored, a definition equivalent to the row-column sum condition can be given which does not mention the elements at all. Let u be the n-dimensional column vector whose components are all equal to $n^{-1/2}$. (We have made $\|u\| = 1$ for convenience.) An $n \times n$ matrix T will have row and column sums one if and only if $Tu = T^*u = u$. In this paper we shall use the term doubly stochastic to describe the members of this larger class of matrices. We label this class $\mathfrak{D}_u$.

In [2] the author gave a characterization of this matrix class in terms of a certain factoring problem:

THEOREM 1. *Let* $1 \oplus 0$ *denote the* n*-dimensional vector whose first component equals one and whose remaining components all equal zero and set*

$$\mathcal{P}_n = \{T \in \mathfrak{M}_n \mid T(1 \oplus 0) = u,\ T'u = 1 \oplus 0\},$$

$\mathfrak{M}_n$ *denoting the* $n \times n$ *complex matrices. Then*

$$\mathfrak{D}_u = \mathcal{P}_n \mathcal{P}_n', \qquad \mathcal{P}_n' \mathcal{P}_n = 1 \oplus \mathfrak{M}_{n-1},$$

and, in fact, if $\mathfrak{X}_n, \mathfrak{Y}_n \subseteq \mathfrak{M}_n$ *are such that* $\mathfrak{D}_u = \mathfrak{X}_n \mathfrak{Y}_n$, $\mathfrak{Y}_n \mathfrak{X}_n = 1 \oplus \mathfrak{M}_{n-1}$, *then there exists a complex number* $\rho \neq 0$ *such that* $\mathfrak{X}_n \subseteq \rho \mathcal{P}_n$, $\mathfrak{Y}_n \subseteq \rho^{-1} \mathcal{P}_n'$. *The inclusion may be proper.* (*A prime on a matrix denotes transpose. A prime on a set of matrices denotes the collection of transposes in that set.*)

It is the intent of this paper to establish results of a similar nature which are not dependent upon the finiteness of dimension. First, then, we extend the notion of doubly stochastic to the infinite-dimensional case.

Let $\mathfrak{H}$ be an infinite-dimensional complex Hilbert space and fix $u \in \mathfrak{H}$, $\|u\| = 1$. The members of

$$\mathfrak{D}_u = \{T \in [\mathfrak{H}] \mid Tu = T^*u = u\}$$

are said to be doubly stochastic on $\mathfrak{H}$.

Presented to the Society, January 24, 1964; received by the editors March 16, 1964.

The following theorem characterizes this generalized $\mathfrak{D}_u$ in terms of a certain set of mappings in $[C\oplus\mathfrak{K}, \mathfrak{K}]$, where C is the complex plane.

THEOREM 2. *Let*

$$\mathcal{P} = \{T \in [C \oplus \mathfrak{K}, \mathfrak{K}] \mid T(1 \oplus 0) = u, \ T^*u = 1 \oplus 0\}.$$

Then $\mathfrak{D}_u = \mathcal{P}\mathcal{P}^A$, $\mathcal{P}^A\mathcal{P} = I_C\oplus[\mathfrak{K}]$, *where* I_C *is the identity function on* C. *In fact, if* $\mathfrak{X}\subseteq[C\oplus\mathfrak{K}, \mathfrak{K}]$ *and* $\mathfrak{Y}\subseteq[\mathfrak{K}, C\oplus\mathfrak{K}]$ *are such that* $\mathfrak{D}_u=\mathfrak{X}\mathfrak{Y}$, $\mathfrak{Y}\mathfrak{X}=I_C\oplus[\mathfrak{K}]$, *then there is a complex number* $\rho\neq 0$ *such that* $\mathfrak{X}\subseteq\rho\mathcal{P}$, $\mathfrak{Y}\subseteq\rho^{-1}\mathcal{P}^A$. *If, in addition,* $\mathfrak{Y}=\mathfrak{X}^A$, *then* $|\rho|=1$. *In any event, the inclusions may be proper.* (*For any collection of mappings,* X, X^A *is used to denote the set of adjoints of the members of* X.)

The first part of Theorem 2 is a consequence of the following two lemmas:

LEMMA 1. $I_C\oplus[\mathfrak{K}]=\{T\in[C\oplus\mathfrak{K}] \mid T(1\oplus 0)=T^*(1\oplus 0)=1\oplus 0\}$.

PROOF. Let $T\in[C\oplus\mathfrak{K}]$ and pick $h\in\mathfrak{K}$. There exist $\mu\in C$, $h'\in\mathfrak{K}$ such that $T(0\oplus h)=\mu\oplus h'$. Then if $T^*(1\oplus 0)=1\oplus 0$,

$$(T(0 \oplus h), 1 \oplus 0) = (0 \oplus h, T^*(1 \oplus 0)) = (0 \oplus h, 1 \oplus 0) = 0,$$

while, at the same time,

$$(T(0 \oplus h), 1 \oplus 0) = (\mu \oplus h', 1 \oplus 0) = \mu.$$

Thus, $\mu=0$, showing that $T(0\oplus h)=0\oplus h'$.

Define T_1: $\mathfrak{K}\to\mathfrak{K}$ by the rule $T_1h=h'$. Clearly, T_1 is linear. Furthermore,

$$\|T_1h\| = \|h'\| = \|0 \oplus h'\| = \|T(0 \oplus h)\| \leqq \|T\| \|0 \oplus h\| = \|T\| \|h\|,$$

showing that T_1 is bounded with $\|T_1\|\leqq\|T\|$. Thus $T_1\in[\mathfrak{K}]$.

If $T(1\oplus 0)=1\oplus 0$, then, for any $\lambda\in C$, $h\in\mathfrak{K}$,

$$\begin{aligned} T(\lambda \oplus h) &= T(\lambda \oplus 0) + T(0 \oplus h) = \lambda T(1 \oplus 0) + (0 \oplus T_1h) \\ &= \lambda(1 \oplus 0) + (0 \oplus T_1h) = \lambda \oplus T_1h = (I_C \oplus T_1)(\lambda \oplus h), \end{aligned}$$

showing that $T=I_C\oplus T_1$.

LEMMA 2. *If* $T_0\in\mathcal{P}$ *is nonsingular, then*

$$\mathfrak{D}_u = T_0\mathcal{P}^A = \mathcal{P}T_0^*;$$
$$T_0^*\mathcal{P} = \mathcal{P}^A T_0 = I_C \oplus [\mathfrak{K}].$$

PROOF. $T\in\mathfrak{D}_u\Rightarrow T_0^{-1}Tu=1\oplus 0$ and $(T_0^{-1}T)^*(1\oplus 0)=T^*T_0^{*-1}(1\oplus 0)=u\Rightarrow T_0^{-1}T\in\mathcal{P}^A\Rightarrow T\in T_0\mathcal{P}^A$. Thus $\mathfrak{D}_u\subseteq T_0\mathcal{P}^A$. Similarly, $\mathfrak{D}_u\subseteq\mathcal{P}T_0^*$.

$S\in I_C\oplus[\mathfrak{K}]\Rightarrow T_0^{*-1}S(1\oplus 0)=u$ and $(T_0^{*-1}S)^*u=S^*T_0^{-1}u=1\oplus 0$

$\Rightarrow T_0^{*-1}S \in \mathcal{P} \Rightarrow S \in T_0^* \mathcal{P}$. Thus $I_C \oplus [\mathcal{H}] \subseteq T_0^* \mathcal{P}$. Similarly, $I_C \oplus [\mathcal{H}] \subseteq \mathcal{P}^A T_0$.

The opposite inclusions are clear.

Lemma 2 and the obvious inclusions, $\mathcal{P}\mathcal{P}^A \subseteq \mathfrak{D}_u$, $\mathcal{P}^A \mathcal{P} \subseteq I_C \oplus [\mathcal{H}]$, give the first part of the theorem.

To demonstrate the remainder of Theorem 2 we analyze the one-dimensional member of $\mathfrak{D}_u$, J. It will be shown, among other things, that J is *unique*.

Since $J \in \mathfrak{D}_u$, $Ju = u$, and since the range of J is one-dimensional, u must generate that range. Then $Jx = \lambda_x u$ for each $x \in \mathcal{H}$, where λ_x is a complex number dependent upon x. But

$$(Jx, u) = (\lambda_x u, u) = \lambda_x(u, u) = \lambda_x,$$

and

$$(Jx, u) = (x, J^*u) = (x, u)$$

showing that, in fact, $Jx = (x, u)u$. In particular, the mapping J is unique.

The following computation shows that J is self adjoint.

$$(Jx, y) = ((x, u)u, y) = (x, u)(u, y),$$
$$(J^*x, y) = (x, Jy) = (x, (y, u)u) = (x, u)(u, y),$$

for all $x, y \in \mathcal{H}$. Furthermore, from

$$(TJx, y) = (T(x, u)u, y) = (x, u)(u, y)$$

and

$$(JTx, y) = (x, T^*J^*y) = (x, T^*Jy) = (x, T^*(y, u)u) = (x, u)(u, y),$$

it is clear that $TJ = JT = J$ for all $T \in \mathfrak{D}_u$.

The next part of Theorem 2 will follow from Lemma 3:

LEMMA 3. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be as in Theorem 2. Then*

$$X[I_C \oplus 0_{\mathcal{H}}] = JX,$$
$$[I_C \oplus 0_{\mathcal{H}}]Y = YJ,$$

for all $X \in \mathfrak{X}, Y \in \mathfrak{Y}$. $0_{\mathcal{H}}$ is the zero operator in $[\mathcal{H}]$.

PROOF. Let $Yx = \mu_x \oplus h_x$, $x \in H$. Then

$$X[I_C \oplus 0_{\mathcal{H}}]Yx = X[I_C \oplus 0_{\mathcal{H}}](\mu_x \oplus h_x) = \mu_x X(1 \oplus 0),$$

showing that $X[I_C \oplus 0_{\mathcal{H}}]\ Y$ is one dimensional. Since $X[I_C \oplus 0_{\mathcal{H}}]Y \in \mathfrak{X}(\mathfrak{Y}\mathfrak{X})\mathfrak{Y} = \mathfrak{D}_u^2 = \mathfrak{D}_u$, it follows that $X[I_C \oplus 0_{\mathcal{H}}]Y = J$. Then, since $YX \in I_C \oplus [\mathcal{H}]$, $YX = I_C \oplus T_1$ for some $T_1 \in [\mathcal{H}]$. Then

$$JX = X[I_C \oplus 0_{\mathfrak{K}}]YX = X[I_C \oplus 0_{\mathfrak{K}}][I_C \oplus T_1] = X[I_C \oplus 0_{\mathfrak{K}}].$$

Similarly, $YJ = [I_C \oplus 0_{\mathfrak{K}}]Y$.

The proof in Theorem 2 may now be completed. Pick $X \in \mathfrak{X}$, $Y \in \mathfrak{Y}$ and let $X^*u = \rho_X \oplus k_X$. With the aid of Lemma 3, $[I_C \oplus 0_{\mathfrak{K}}]X^*u = X^*Ju = X^*u$, i.e.,

$$\rho_X \oplus k_X = [I_C \oplus 0_{\mathfrak{K}}](\rho_X \oplus k_X) = \rho_X \oplus 0.$$

Thus $X^*u = \rho_X \oplus 0 = \rho_X(1 \oplus 0)$. In the same way, $Yu = \rho_Y(1 \oplus 0)$, where ρ_Y may depend upon Y.

Since $XY \in \mathfrak{D}_u$,

$$(X^*u, Yu) = (u, XYu) = (u, u) = 1.$$

But

$$(X^*u, Yu) = (\rho_X(1 \oplus 0), \rho_Y(1 \oplus 0)) = \rho_X\bar{\rho}_Y$$

and, thus, $\rho_X\bar{\rho}_Y = 1$. By letting X vary over $\mathfrak{X}$ and holding Y fixed, we conclude that $\rho_X = \bar{\rho}$, a constant for all X. Then $\rho_Y = \rho^{-1}$ for all Y. Of course, $\rho \neq 0$.

Since $X^*u = \bar{\rho}(1 \oplus 0)$ and $(XY)^* \in \mathfrak{D}_u^A = \mathfrak{D}_u$,

$$Y^*(1 \oplus 0) = \bar{\rho}^{-1}Y^*X^*u = \bar{\rho}^{-1}u.$$

Similarly, $X(1 \oplus 0) = \rho XYu = \rho u$. It follows that $X \in \rho\mathcal{P}$, $Y \in \rho^{-1}\mathcal{P}^A$, i.e., that $\mathfrak{X} \subseteq \rho\mathcal{P}$, $\mathfrak{Y} \subseteq \rho^{-1}\mathcal{P}^A$. If it happens that $\mathfrak{Y} = \mathfrak{X}^A$, then $\bar{\rho} = \rho^{-1}$ and we must have $|\rho| = 1$.

The inclusions may be proper. Suppose $\mathcal{P}$ contains three distinct nonsingular elements T_1, T_2, and T_3. Define $\mathfrak{X} = \mathcal{P} - \{T_1\}$. By Lemma 2, $T_2\mathfrak{X}^A = \mathfrak{D}_u - \{T_2T_1^*\}$. But, since $T_3 \neq T_2$, $T_3^{-1}T_2T_1^* \neq T_1^*$ and it follows that $T_3^{-1}T_2T_1^* \in \mathfrak{X}^A = \mathfrak{Y}$. Whence $T_3(T_3^{-1}T_2T_1^*) = T_2T_1^* \in \mathfrak{X}\mathfrak{Y}$.

Similarly, $T_2^*\mathfrak{X} = \{I_C \oplus [\mathfrak{K}]\} - \{T_2^*T_1\}$ and, since $T_3^{*-1}T_2^*T_1 \in \mathfrak{X}$, it follows that $T_3^*(T_3^{*-1}T_2^*T_1) = T_2^*T_1 \in \mathfrak{Y}\mathfrak{X}$. Thus $\mathfrak{D}_u = \mathfrak{X}\mathfrak{Y}$ and $\mathfrak{Y}\mathfrak{X} = I_C \oplus [\mathfrak{K}]$, even though $\mathfrak{X} \subset \mathcal{P}$ and $\mathfrak{Y} \subset \mathcal{P}^A$ properly.

References

1. Marvin Marcus and Henryk Minc, *A survey of matrix theory and matrix inequalities*, Allyn and Bacon, Boston, 1964.

2. Richard Sinkhorn, *On the factor spaces of the complex doubly stochastic matrices*, Abstract 62T-243, Notices Amer. Math. Soc. 9 (1962), 334–335.

University of Houston