A NEW PROOF OF THE BONNICE-KLEE THEOREM

JOHN R. REAY

The purpose of this note is to give a new (and much easier) proof for a theorem of Bonnice-Klee [1]. Their result (Theorem 1 below) has proved to be quite useful in establishing a number of results concerning the convex hulls of certain sets (see [4]). It is a generalization of the following two classic results due respectively to Carathéodory and Steinitz.

(a) If X is a subset of an n-dimensional real linear space E^n, and $p \in \operatorname{con} X$ (the convex hull of X), then $p \in \operatorname{con} U$ for some subset U of X with card U (cardinality of U) at most $n+1$.

(b) If $X \subset E^n$, and $p \in \operatorname{int} \operatorname{con} X$ (the interior of con X), then $p \in \operatorname{int} \operatorname{con} U$ for some subset U of X with card $U \leqq 2n$.

The generalization is based on the notion of intermediate interiors. The *d-interior* of a set $X \subset E^n$ (denoted by $\operatorname{int}_d X$) is the set of all points p such that p is in the relative interior of some d-simplex contained in X; equivalently, $p \in \operatorname{int}_d X$ iff there exists a d-dimensional flat F through p such that p is in the interior of $X \cap F$ relative to F. The result may be stated as

THEOREM 1. *If* $X \subset E^n$, $0 \leqq d \leqq n$, *and* $p \in \operatorname{int}_d \operatorname{con} X$, *then* $p \in \operatorname{int}_d \operatorname{con} U$ *for some subset* U *of* X *with* card $U \leqq \max(n+1, 2d)$.

Theorem 2 below was used in [4] as a tool to establish certain generalizations of the above result of Steinitz, and has also been used to obtain uniquely-defined continuous representations of points in E^n in terms of an arbitrary positive basis. We will show that Theorem 1 of Bonnice-Klee is an easy consequence of Theorem 2. Set $B \subset E^n$ *positively spans* E^n if each point of E^n is a positive combination (i.e., a linear combination with non-negative coefficients) of the points of B. The set B is a *positive basis* of E^n if it is also *positively independent*, i.e., no element of B is a positive combination of the remaining elements of B. (See [2], [3].) The *positive hull* of set X, denoted by pos X, is the set of all positive combinations of X.

THEOREM 2. *Let* B *be any positive basis for* E^n. *Then* B *admits a partition into pair-wise disjoint subsets* $B = B_1 \cup \cdots \cup B_k$ $(1 \leqq k \leqq n)$, *such that* card $B_i \geqq$ card $B_{i+1} \geqq 2$, $i = 1, \cdots, k-1$, *and* $\operatorname{pos}(B_1 \cup \cdots \cup B_j)$ *is a linear subspace of* E^n *of dimension* $(\sum_{i=1}^{j} \operatorname{card} B_i) - j$ *for* $j = 1, 2, \cdots, k$.

Presented to the Society, June 20, 1964; received by the editors May 16, 1964.

We will sketch the proof of Theorem 2, since certain parts of the machinery are needed in proving Theorem 1. It is well known that if B is a positive basis for an n-dimensional space E, then $n+1 \leqq \text{card } B \leqq 2n$. Set B is said to be *minimal* provided that card $B=n+1$. A subset B' of an arbitrary positive basis B is in general not a positive basis for a linear subspace. Following the notation of McKinney [3], we say that a linear subspace F of E is a *spanned subspace with respect to* B iff $F \cap B$ is a positive basis for F. If, moreover, $F \cap B$ is a minimal positive basis for F, then we say that F is a *minimal subspace* (with respect to B). Davis [2] has shown that for each element b of a positive basis B, there exists a set B' (not necessarily unique) such that $b \in B' \subset B$ and pos B' is a minimal subspace of E with respect to B. Thus E always has at least one minimal subspace with respect to B.

Proof of Theorem 2. Let B be a given positive basis for the space E, and let B_1 be a subset of B of maximal cardinality such that B_1 is a minimal positive basis for the minimal subspace pos B_1. If $B=B_1$, the theorem is clear. Otherwise, let E_1 be a linear subspace of E such that $E=E_1 \oplus \text{pos } B_1$ ($\oplus$ means direct linear sum), and let π_1 be the natural projection of E onto E_1. It follows that $\pi_1(B-B_1)$ is a positive basis for E_1. Also, if B_2 is a subset of $B-B_1$ of maximal cardinality such that $\pi_1 B_2$ is a minimal positive basis for the minimal subspace pos $\pi_1 B_2$ of E_1, then it follows that card $B_1 \geqq$ card B_2 and $\text{pos}(B_1 \cup \pi_1 B_2) = \text{pos}(B_1 \cup B_2)$. (See [4] for details of these arguments.) It is clear that card $B_1 \geqq$ card $B_2 \geqq 2$ and $\text{pos}(B_1 \cup B_2)$ is a linear space of dimension card B_1+card B_2-2. If $B_1 \cup B_2 = B$ the theorem is established. Otherwise the same argument may be applied to the space E_1 and the process repeated until $B=B_1 \cup \cdots \cup B_k$, thus establishing Theorem 2.

Proof of Theorem 1. Suppose $X \subset E^n$, $0 \leqq d \leqq n$, and $p \in \text{int}_d \text{con } X$. With no loss of generality we may suppose that $p=0$. Let F denote the largest linear space contained in pos X, and let m be the dimension of F. Then $m \geqq d$ because $0 \in \text{int}_d \text{con } X$. It follows from the maximality of m that $(\text{con } X) \cap F = \text{con}(X \cap F)$. Thus we may restrict our attention to the set $X \cap F$ in the linear space F, or equivalently, we may assume that pos $X=E^n$. That is, X positively spans E^n. Choose a subset B of X that is a positive basis for E^n, and let $B=B_1 \cup \cdots \cup B_k$ be a partition of B as in Theorem 2. Then B_1 is a minimal positive basis in B of maximal cardinality. Thus card $B_1 \leqq n+1$. If $d<$card B_1 then $0 \in \text{int}_d \text{con } B_1$ and we set $U=B_1$, establishing Theorem 1. For the case of $d \geqq$ card B_1, let $U=B_1 \cup B_2 \cup \cdots \cup B_j$ where j is the least integer such that the dimension of the linear space $\text{pos}(B_1 \cup B_2 \cup \cdots \cup B_j)$ is at least d. It is clear that $0 \in \text{int}_d \text{con } U$,

and it only remains to show that card $U \leqq \max(n+1, 2d)$. Note that $d > \dim \operatorname{pos}(B_1 \cup \cdots \cup B_{j-1}) = (\sum_{i=1}^{j-1} \operatorname{card} B_i) - (j-1)$. Thus card $U = \sum_{i=1}^{j} \operatorname{card} B_i < d + (j-1) + \operatorname{card} B_j$ and it suffices to show that $(j-1) + \operatorname{card} B_j \leqq d+1$. But

$$d \geqq \left(\sum_{i=1}^{j-1} \operatorname{card} B_i \right) - (j-1) + 1 \geqq \operatorname{card} B_1 + 2(j-2) - (j-1) + 1$$

$$= (j-1) + \operatorname{card} B_1 - 1.$$

So $d+1 \geqq (j-1) + \operatorname{card} B_j$, which shows that card $U \leqq 2d$ and thus establishes Theorem 1.

References

1. W. Bonnice and V. L. Klee, *The generation of convex hulls*, Math. Ann. **152** (1963), 1–29.

2. C. Davis, *Theory of positive linear dependence*, Amer. J. Math. **76** (1954), 733–746.

3. R. L. McKinney, *Positive bases for linear spaces*, Trans. Amer. Math. Soc. **103** (1962), 131–148.

4. J. R. Reay, *Generalizations of a theorem of Carathéodory*, Mem. Amer. Math. Soc. No. 54 (1965).

Western Washington State College