CELLULAR SUBCOMPLEXES OF PIECEWISE-LINEAR MANIFOLDS[1,2]

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Let K be a finite cellular subcomplex of a piecewise-linear (combinatorial) n-manifold, M^n, and let N be a regular neighborhood of K in M^n.

LEMMA 1. *K is contractible.*

PROOF. That K is homologically trivial is an easy application of Čech homology theory. We prove $\pi_1(K)=1$. There is an n-cell B such that $K\subset B^0\subset B\subset N$. There is a retraction $\phi: N\twoheadrightarrow K$. Let $\psi=\phi|B$. Then $\psi_\#: \pi_1(B)\to\pi_1(K)$ is an epimorphism. Since $\pi_1(B)=1$ we have $\pi_1(K)=1$. The proof is completed by an appeal to the Hurewicz theorem.

LEMMA 2. *∂N is a homotopy $(n-1)$-sphere, $n\geqq 3$.*

PROOF. Since K is contractible, so is N. Thus, if x_0 is any point of N^0, $\pi_1(N-x_0)=1$. $N-x_0$ is homeomorphic to $N-K$ and there is a retraction of $N-K$ onto ∂N. Hence, $1=\pi_1(N-x_0)=\pi_1(N-K)=\pi_1(\partial N)$.

By giving a duality argument similar to that found in [3, Proposition I.1, p. 152] we can prove that ∂N is an homology $(n-1)$-sphere. Consequently, the proof is complete.

Let us denote the n-dimensional generalized Poincaré conjecture by PC(n) (see [4], [5], and [7] for proofs) and let Cell(n) denote the conjecture: N is an n-cell. Then we have the following theorems, the proof of the first of which is essentially that given for Proposition 1 of [2].

THEOREM 1. *$PC(n-1)+PC(n)$ implies Cell(n).*

THEOREM 2. *Cell(n)$+PC(n)$ implies $PC(n-1)$, $n\geqq 5$.*

PROOF. Let Σ^{n-1} be a piecewise-linear closed $(n-1)$-manifold which is a homotopy $(n-1)$-sphere. Let B be an $(n-1)$-simplex in Σ and let $F=\Sigma-B^0$. F is a contractible $(n-1)$-manifold whose boundary is

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an $(n-2)$-sphere. All we need demonstrate is that F is an $(n-1)$-cell.

Since F is contractible, $F\times I$ is also. Hence, $2(F\times I)$ is a homotopy n-sphere so, by hypothesis, $2(F\times I)$ is an n-sphere.

Now $2(F\times I)-\partial(F\times I)=D_1\cup D_2$ where D_i is $(F\times I)^0$. D_1 is contractible and 1-connected at infinity and dimension of $D_1=n\geqq 5$. Consequently, $D_1=E^n$ [6]. Thus, $\overline{D}_2=F\times I$ is cellular in $2(F\times I)=S^n$ and the regular neighborhood of $F\times I$ in $2(F\times I)$ is, by hypothesis, an n-cell. However, this regular neighborhood is homeomorphic to $F\times I$. Hence, $F\times I=I^n$ and $\partial(F\times I)=2F=S^{n-1}$. By the generalized Schoenflies theorem [1], F is an $(n-1)$-cell.

References

1. M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.

2. M. L. Curtis, *Cartesian products with intervals*, Proc. Amer. Math. Soc. **12** (1961), 819–820.

3. M. L. Curtis and R. L. Wilder, *The existence of certain types of manifolds*, Trans. Amer. Math. Soc. **91** (1959), 152–160.

4. S. Smale, *The generalized Poincaré conjecture in dimensions greater than four*, Ann. of Math. **74** (1961), 391–406.

5. J. Stallings, *Polyhedral homotopy spheres*, Bull. Amer. Math. Soc. **66** (1960), 485–488.

6. ———, *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488.

7. E. C. Zeeman, *The generalized Poincaré conjecture*, Bull. Amer. Math. Soc. **67** (1961), 270.

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