IRRATIONAL POWER SERIES. III[1]

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Let α be an irrational number and β a real number. Denote by $\{y\}$ the fractional part of y and so $0 \leqq \{y\} < 1$. Let the function $f(y)$ be defined for $0 \leqq y < 1$.

Let $F(x)$, for $|x| < 1$, be defined by the series

$$F(x) = \sum_{n=0}^{\infty} f(\{\alpha n + \beta\})x^n. \tag{1}$$

Several writers (see [1] for references) have shown that if $f(y)$ satisfies various conditions, then $F(x)$ is not a rational function of x, and some have shown that the circle $|x| = 1$ is a line of essential singularities for $F(x)$. I notice a very simple way of dealing with such questions. The method is really that of Hecke, but its possibilities have not been fully explored. I now prove the

THEOREM. *Let $f(x)$ be Riemann integrable in $0 \leqq x \leqq 1$, and let l be any integer. Then as $x \to e^{2l\pi i\alpha}$ along a radius from $x = 0$,*

$$F(x)(1 - xe^{-2l\pi i\alpha}) \to e^{-2l\pi i\beta} \int_0^1 f(t)e^{2l\pi it}\, dt. \tag{2}$$

Further, if $f(x)$ is continuous except for a finite number of finite discontinuities for $0 \leqq x < 1$, $F(x)$ is a rational function of x if and only if $f(x)$ is a finite Fourier series,

$$f(x) = \sum_{r=-L}^{L} a_r e^{2r\pi ix}, \tag{3}$$

and then

$$F(x) = \sum_{r=-L}^{L} \frac{a_{-r}e^{2r\pi i\beta}}{1 - xe^{2r\pi i\alpha}}. \tag{4}$$

If $f(x)$ is not of the form (3), *then $F(x)$ cannot be continued outside of of the circle $|x| = 1$.*

We require two lemmas of which the first is well known.

LEMMA 1. *If the series $g(x) = \sum_0^\infty a_n x^n$ converges for $|x| < 1$ and if*

Presented to the Society, April 18, 1964; received by the editors March 18, 1964.

[1] This work was supported in part by the National Science Foundation.

(5) $$\lim_{N\to\infty} (a_0 + a_1 + \cdots + a_N)/N = k,$$

then, as $x\to 1-0$,

(6) $$(1 - x)g(x) \to k.$$

LEMMA 2 (WEYL) [2]. *If* $g(n)$ *is uniformly distributed* mod 1 *for* $n=1, 2, 3, \cdots$, *and* $w(t)$ *is Riemann integrable in* $0\leqq t\leqq 1$, *then*

(7) $$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} w(\{g(n)\}) = \int_0^1 w(t)\,dt.$$

In (1), put $x=Xe^{2l\pi i\alpha}$, then

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} f(\{\alpha n + \beta\})e^{2ln\pi i\alpha}X^n \\ &= e^{-2\pi il\beta}\sum_{n=1}^{\infty} f(\{\alpha n + \beta\})e^{2\pi il\{\alpha n+\beta\}}x^n. \end{aligned}$$

Hence, from (7), with $g(n)=\alpha n+\beta$ and $w(t)=f(t)e^{2l\pi it}$,

(8) $$F(x)(1 - xe^{-2l\pi i\alpha}) \to e^{-2l\pi i\beta}\int_0^1 f(t)e^{2l\pi it}\,dt.$$

Clearly $x=e^{2l\pi i\alpha}$ is a singularity of $F(x)$ if $\int_0^1 f(t)e^{2l\pi it}\,dt \neq 0$. Then $F(x)$ can be a rational function of x if and only if $\int_0^1 f(t)e^{2l\pi it}\,dt=0$ for all integers l except a finite number of values. Since $f(x)$ is such that for $0\leqq x\leqq 1$, the Fourier coefficients of $f(x)$ exist, we have the usual association of $f(x)$ with its Fourier series given by

(9) $$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2n\pi ix} = \lim_{N\to\infty}\sum_{n=-N}^{N} a_n e^{2n\pi ix}.$$

It is known [3] that $\int_0^1 f(t)e^{2l\pi it}\,dt$ can be evaluated by term-by-term integration on substituting for $f(t)$ from (9). If the integral is zero, $f(t)$ cannot have a term $e^{-2l\pi it}$ in (9), and then (3) follows.

It is also clear that, unless (3) holds, the integral cannot vanish for all large l. Then, since $\{l\alpha\}$ is uniformly distributed mod 1, $|x|=1$ is a line of essential singularities for $F(x)$.

This result also holds if in (9), $a_n=0$ except when n belongs to an arithmetical progression.

The theorem is easily extended to more general series in which f is a function of several variables. Similar results hold.

(10) $$F(x) = \sum_{n=0}^{\infty} f(\{\alpha_1 n + \beta_1\}, \{\alpha_2 n + \beta_2\})x^n,$$

where α_1, α_2 are irrational numbers and β_1, β_2 are real numbers. We suppose α_1, α_2, 1 are linearly independent over the rationals and then the points $(\{\alpha_1 n+\beta_1\}, \{\alpha_2 n+\beta_2\})$, for $n=1, 2, 3, \cdots$, are uniformly distributed in the unit square. This follows [4] since

$$\sum_{n=1}^{N} e^{2\pi i((\alpha_1 n+\beta_1)h_1+(\alpha_2 n+\beta_2)h_2)} = o(N)$$

for all fixed integers h_1, h_2 not both zero. Lemma 2 is replaced by

LEMMA 3. *If* $(\{g_1(n)\}, \{g_2(n)\})$ *is uniformly distributed in the unit square, then if* $w(t_1, t_2)$ *is Riemann integrable in* $0 \leqq t_1, t_2 \leqq 1$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} w(\{g_1(n)\}, \{g_2(n)\}) = \int_0^1 \int_0^1 w(t_1, t_2)\, dt_1 dt_2. \tag{11}$$

In (10), put $x=e^{2\pi i l_1\alpha_1+2\pi i l_2\alpha_2}X$, where l_1, l_2 are any integers. The result corresponding to (8) is

$$(1 - xe^{-2\pi l_1\alpha_1-2\pi i l_2\alpha_2})F(x) \to e^{-2\pi i l_1\beta_1-2\pi i l_2\beta_2} \int_0^1 \int_0^1 f(t_1, t_2)\, dt_1 dt_2. \tag{12}$$

We can show as before that if $f(x, y)$ is continuous for $0 \leqq x, y < 1$, then $F(x)$ is a rational function of x if, and only if, $f(x, y)$ is a finite Fourier series $\sum_{r,s=-L}^{L} a_{r,s} e^{2\pi i(rx+sy)}$. If this does not hold, $F(x)$ cannot be continued outside $|x|=1$.

In conclusion, it may be noted that meromorphic expansions of (1) can be found [1] by substituting for $f(y)$ its Fourier expansion and inverting the order of summation. The justification may be rather delicate and poses a difficult problem.

REFERENCES

1. L. J. Mordell, *Irrational power series*. II, Acta Arith. (to appear).
2. J. F. Koksma, *Diophantische Approximationen*, Springer, Berlin, 1936; p. 91.
3. E. C. Titchmarsh, *Functions of a complex variable*, Clarendon Press, Oxford, 1957.
4. J. F. Koksma, *Diophantische Approximationen*, Springer, Berlin, 1936; p. 92.

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