THE NUMBER OF COPRIME CHAINS WITH LARGEST MEMBER n

R. C. ENTRINGER

1. In a previous paper [1] a *coprime chain* was defined to be an increasing sequence $\{a_1, \cdots, a_k\}$ of integers greater than 1 which contains exactly one multiple of each prime equal to or less than a_k.

We let $s(n)$, $n>1$, denote the number of coprime chains with largest member n. For convenience we define $s(1)=1$.

In this paper we will obtain a partial recursion formula for $s(n)$ and an asymptotic formula for $\log s(n)$. A table of values of $s(n)$, $n \leqq 113$, is also provided.

In the following p will designate a prime and p_i will designate the ith prime.

2. LEMMA 1. *$A=\{a_1, \cdots, a_k=p_i \neq 2\}$ is a coprime chain iff*

(i) *$A'=\{a_1, \cdots, a_{k-1}\}$ is a coprime chain,*

(ii) *p_{i-1} is the largest prime in A'.*

PROOF. If $A=\{a_1, \cdots, a_k=p_i \neq 2\}$ is a coprime chain, then

(ii) p_{i-1} is in A (and therefore is the largest prime in A') since by Bertrand's Postulate $2p_{i-1}>p_i$, and

(i) If A' is not a coprime chain, then there is a prime $p \leqq a_{k-1}$ dividing no member of A'. Thus p divides (and therefore is equal to) a_k since A is a coprime chain, but this is impossible since $a_{k-1}<a_k$.

To prove the converse we note that if A is not a coprime chain, then p_i divides some member of A' and therefore $p_{i-1}<a_{k-1}/2$. But again by Bertrand's Postulate there is a prime between $a_{k-1}/2$ and a_k occurring in A' which contradicts (ii).

A direct result of this lemma is:

THEOREM 2. $s(p_i)=\sum_{n=p_{i-1}}^{p_i-1} s(n)$, $i \geqq 2$.

THEOREM 3. $s(p)=\sum_{n<p} s(n)$ (*n not prime*).

PROOF. The assertion holds for $p=2$. Now let q and p be successive primes with $q<p$. If $s(q)=\sum_{n<q} s(n)$ (n not prime), then

$$s(p) = s(q) + \sum_{q<n<p} s(n) = \sum_{n<p} s(n) \quad (n \text{ not prime})$$

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by Theorem 2 and the theorem follows by induction.

3. The above result indicates marked irregularities in $s(n)$, however, we can approximate $\log s(n)$ asymptotically.

THEOREM 4. $\log s(n) \sim \sqrt{n}$.

PROOF. Every coprime chain $A(n)$ can be constructed in the following manner. Let q_i, $i=1, \cdots, k$, $q_i > q_j$ for $i<j$ be those primes less than $\sqrt{n}$ and not dividing n. Choose any multiple m_1q_1 of q_1 so that $m_1q_1 \leqq n$ and $(m_1, n)=1$. If $q_2 \mid m_1$ let $m_2=0$. If $q_2 \nmid m_1$, choose any multiple m_2q_2 of q_2 so that $m_2q_2 \leqq n$ and $(m_2, nm_1q_1)=1$. This process is continued by choosing $m_i=0$ if $q_i \mid m_j$ for some $j=1, \cdots, i-1$, otherwise choosing any multiple m_iq_i of q_i so that $m_iq_i \leqq n$, $(m_i, nm_1q_1 \cdots m_{i-1}q_{i-1})=1$. The set $\{m_1q_1, \cdots, m_kq_k\} - \{0\}$ can then be extended to a coprime chain by appending n and those primes p between $\sqrt{n}$ and n which do not divide n or any m_i, and reordering if necessary. This extension is unique since any multiple of a prime p, other than p itself, must either be larger than n, not relatively prime to n, or not relatively prime to all m_iq_i. Therefore

$$\log s(n) \leqq \log_{p \leqq \sqrt{n}} \left[\frac{n}{p}\right] \leqq \sum_{p \leqq \sqrt{n}} \log n - \sum_{p \leqq \sqrt{n}} \log p = \{1 + o(1)\}\sqrt{n}.$$

To obtain a lower bound for $\log s(n)$, coprime chains are constructed by choosing the m_i in the following manner. Let m_1 be 1 or any prime satisfying $\sqrt{n} < m_1 \leqq n/q_1$, $m_1 \nmid n$. There are at least $\pi(n/q_1) - \pi(\sqrt{n}) - 1$ choices for m_1 since there is at most one prime in the given range which divides n. Let m_2 be 1 or any prime satisfying $\sqrt{n} < m_2 \leqq n/q_2$, $m_2 \nmid nm_1$. There are at least $\pi(n/q_2) - \pi(\sqrt{n}) - 2$ choices for m_2. This process is continued until all multiples m_iq_i have been chosen. In general there are at least

$$\begin{aligned} \pi\left(\frac{n}{q_i}\right) - \pi(\sqrt{n}) - i &\geqq \pi\left(\frac{n}{q_i}\right) - \pi(\sqrt{n}) - \{\pi(\sqrt{n}) - \pi(q_i)\} \\ &= \pi\left(\frac{n}{q_i}\right) - 2\pi(\sqrt{n}) + \pi(q_i) \end{aligned}$$

choices for m_i. The set $\{m_1q_1, \cdots, m_kq_k\}$ is then extended to a coprime chain as previously indicated. If $\pi(n/q_i) - 2\pi(\sqrt{n}) + \pi(q_i) \leqq 0$, then m_i is chosen to be 1; hence the above construction is valid.

In the remainder of the proof we assume ϵ given such that $0 < \epsilon < 1/2$. Define δ by $n^\delta/\delta = 2(1-\epsilon)\sqrt{n}$, $1/\log n < \delta < 1/2$. Then using certain results from [2] we have

$$\log s(n) \geqq \sum_{p \leqq n^\delta;\, p \neq n} \log \left\{ \pi\left(\frac{n}{p}\right) - 2\pi(\sqrt{n}) + \pi(p) \right\}$$

$$\geqq \sum_{17 \leqq p \leqq n^\delta} \log \left\{ \frac{n}{p \log \frac{n}{p}} - \frac{4\sqrt{n}}{\log n - 3} + \frac{p}{\log p} \right\} - \sum_{p \mid n} \log 2n$$

$$= \sum_{p \leqq n^\delta} \log \frac{n}{p \log \frac{n}{p}}$$

$$+ \sum_{p \leqq n^\delta} \log \left\{ 1 - \left(\frac{4\sqrt{n}}{\log n - 3} - \frac{p}{\log p} \right) \frac{p}{n} \log \frac{n}{p} \right\} + o(\sqrt{n})$$

provided that

$$\frac{n}{p \log \frac{n}{p}} - \frac{4\sqrt{n}}{\log n - 3} + \frac{p}{\log p} > 0 \quad \text{for } p \leqq n^\delta. \tag{1}$$

Now for sufficiently large n

$$\sum_{p \leqq n^\delta} \log \frac{n}{p \log \frac{n}{p}} = \{1 + o(1)\} \left(\frac{n^\delta}{\delta} - n^\delta \right) + o(\sqrt{n}),$$

$$= \{1 + o(1)\} 2(1 - \delta)(1 - \epsilon)\sqrt{n} \geqq (1 - \epsilon)^2 \sqrt{n};$$

hence it remains only to show (1) and

$$- \sum_{p \leqq n^\delta} \log \left\{ 1 - \left(\frac{4\sqrt{n}}{\log n - 3} - \frac{p}{\log p} \right) \frac{p}{n} \log \frac{n}{p} \right\} = o(\sqrt{n}).$$

Noting that $p \log (n/p)$ and $p^2(1 - \log n/\log p)$ are increasing functions of p for $p \leqq \sqrt{n}$ and n sufficiently large we have

$$\left(\frac{4\sqrt{n}}{\log n - 3} - \frac{p}{\log p} \right) p \log \frac{n}{p} = \frac{4\sqrt{n}}{\log n - 3} p \log \frac{n}{p} + p^2 \left(1 - \frac{\log n}{\log p} \right)$$

$$\leqq \frac{4\sqrt{n}}{\log n - 3} n^\delta (1 - \delta) \log n + n^{2\delta} \left(1 - \frac{1}{\delta} \right)$$

$$= 4(1 - \delta)(1 - \epsilon)\delta n \left(\frac{2 \log n}{\log n - 3} - 1 + \epsilon \right)$$

$$\leqq (1 - \epsilon)\, n(2 + \epsilon^2 - 1 + \epsilon) = (1 - \epsilon^3) n$$

for all sufficiently large n. Hence (1) holds and

$$\sum_{p\le n^\delta} \log \left\{1 - \left(\frac{4\sqrt{n}}{\log n - 3} - \frac{p}{\log p}\right)\frac{p}{n}\log\frac{n}{p}\right\}$$

$$\ge \sum_{p\le n^\delta} 3\log \epsilon \ge 8\,\frac{\sqrt{n}}{\log n}\log\epsilon$$

which completes the proof.

n	$s(n)$	$\frac{e^{\sqrt{n}}}{s(n)}$	n	$s(n)$	$\frac{e^{\sqrt{n}}}{s(n)}$	n	$s(n)$	$\frac{e^{\sqrt{n}}}{s(n)}$
2	1	4.11	40	6		77	391	
3	1	5.65	41	212	2.84	78	9	
4	1		42	2		79	2005	3.61
5	1	3.83	43	214	3.29	80	25	
6	1		44	15		81	228	
7	3	4.73	45	12		82	117	
8	1		46	19		83	2375	3.81
9	3		47	260	3.65	84	4	
10	2		48	3		85	447	
11	9	3.06	49	154		86	142	
12	1		50	11		87	292	
13	10	3.68	51	62		88	91	
14	2		52	31		89	3351	3.73
15	4		53	521	2.78	90	3	
16	3		54	5		91	715	
17	19	3.25	55	129		92	175	
18	1		56	19		93	392	
19	20	3.80	57	90		94	213	
20	2		58	54		95	826	
21	6		59	818	2.64	96	23	
22	4		60	2		97	5698	3.32
23	32	3.79	61	820	3.03	98	65	
24	1		62	54		99	312	
25	21		63	44		100	47	
26	7		64	57		101	6122	3.78
27	16		65	207		102	19	
28	7		66	7		103	6141	4.16
30	1		67	1189	3.01	104	166	
31	85	3.08	68	62		105	24	
32	9		69	147		106	269	
33	18		70	8		107	6600	4.28
34	11		71	1406	3.24	108	23	
35	35		72	9		109	6623	5.16
36	3		73	1415	3.63	110	31	
37	161	2.72	74	80		111	540	
38	15		75	37		112	76	
39	30		76	73		113	7270	5.69

4. The table on the preceding page lists the value of $s(n)$ for all $n \leqq 113$. All entries for $s(n)$ were computed individually and checked by means of Theorem 2.

References

1. R. C. Entringer, *Some properties of certain sets of coprime integers*, Proc. Amer. Math. Soc. **16** (1965), 515–521.

2. J. B. Rosser and Lowell Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.

University of New Mexico

ON THE CONTENT OF POLYNOMIALS

FRED KRAKOWSKI

1. **Introduction.** The content $C(f)$ of a polynomial f with coefficients in the ring R of integers of some algebraic number field K is the ideal in R generated by the set of coefficients of f. This notion plays an important part in the classical theory of algebraic numbers. Answering a question posed to the author by S. K. Stein, we show in the present note that content, as a function on $R[x]$ with values in the set J of ideals of R, is characterized by the following three conditions:

(1) $C(f)$ depends only on the *set* of coefficients of f;

(2) if f is a constant polynomial, say $f(x)=a$, $a \in R$, then $C(f) = (a)$, where (a) denotes the principal ideal generated by a;

(3) $C(f \cdot g) = C(f) \cdot C(g)$ (Theorem of Gauss-Kronecker, see [1, p. 105]).

2. **Characterization of content.** Denote by $[f]$ the set of nonzero coefficients of $f \in R[x]$ and call f, g equivalent, of $f \sim g$, if $[f]=[g]$. A polynomial is said to be primitive if its coefficients are rational integers and if the g.c.d. of its coefficients is 1.

Lemma. *Let S be a set of polynomials with coefficients in R and suppose it satisfies*:

(1) $1 \in S$;

(2) *if $f \in S$ and $f \sim g$, then $g \in S$;*

(3) *if $f \cdot g \in S$, then $f \in S$ and $g \in S$.*

Then S contains all primitive polynomials.

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