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A NOTE ON COMPACT TRANSFORMATION GROUPS WITH A FIXED END POINT

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Dedicated to A. D. Wallace for his 60th birthday

1. Introduction. In [**2**], Professor A. D. Wallace proved the following: "Let T be a cyclic transformation group of a Peano continuum X leaving fixed an end point, then T has another fixed point." In [**4**], Professor H. C. Wang arrived at the same result by assuming that T is compact and X is an arcwise connected Hausdorff space. In this note, under the same assumption as Wang's, we prove that T has countably many fixed points. In fact, we prove the following

Theorem. *Let (X, T, π) be a transformation group where X is an arcwise connected Hausdorff space. Let A be a closed T-invariant set which is separated from any other closed T-invariant set B, $B\cap A=\emptyset$, by a point. If there is such a closed set B, then T has at least two distinct fixed points, one of them contained in A. If, furthermore, every orbit, under T, is closed, then T has countably many fixed points.*

2. Proof of the theorem. The main technique of the proof is based on the proof used in [**4**] with some modification. Choose $a\in A$ and $b\in B$; connect a and b by an arc $l(t)$, $0\leqq t\leqq 1$, with $l(0)=a$ and $l(1)=b$. Let S be the set of all points which separate A and B. Then, by our assumption, S is not empty. It is clear that S lies on the arc $l(t)$, as does $\mathrm{cl}(S)$. It is also obvious that $g(S)=S$ and $g(\mathrm{cl}(S))=\mathrm{cl}(S)$ for

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all $g \in T$. Introduce a linear ordering in $l(t)$, $0 \leqq t \leqq 1$, by the natural linear ordering of t.

Let $p = l(t_0)$ be the lower limit of S. Then $p \in \mathrm{cl}(S)$. We show $p = g(p)$ for all $g \in T$. Suppose $g(p) \neq p$ for some $g \in T$, then $g(p) > p$ and $g^{-1}(p) > p$. If $p \notin S$, there is a $c \in S$ sufficiently close to p so that $g(c) > c$ and $g^{-1}(c) > c$. If $p \in S$, we choose $c = p$. Let $L = \{ l(t) \mid a \leqq l(t) \leqq c \}$. In either case $A \cup L$ is not separated by $g^{-1}(c)$, but $g(A \cup L)$ is separated by $c = g(g^{-1}(c))$—a contradiction! Consequently, we have $g(p) = p$ for all $g \in T$.

In a similar manner, we can prove that the upper limit $q = l(t_2)$ of S is T-invariant.

We show $p \in A$. Suppose not. Let $B' = B \cup \{p\}$; then B' is a closed T-invariant set with $B' \cap A = \emptyset$. By assumption, there is a point d which separates A and B'. It is clear that d lies on the arc $l(t)$, $0 \leqq t \leqq t_0$, and $d < p$. Since d separates A and B also, $d \in S$. This is a contradiction of the fact that p is the lower limit of S.

Suppose $p = q$ then S contains only one point and $S \in A$. A contradiction! Hence $p \neq q$.

Let every orbit under T be closed. Choose $s = l(t_3) \in S$, with $s < q$. Then $T(s)$ is closed and $T(s) \subset S$. Hence $T(s) \cap A = \emptyset$. Let S' be the nonempty set which contains all the points which separate A and $T(s)$. Again S' lies on the arc $l(t)$, $0 \leqq t \leqq t_3$. By the same method used before, we can prove that the upper limit point p_1 of S', which is different from both p and q, is fixed under T. Continuing this procedure, we arrive at the desired conclusion that T has countably many fixed points. The theorem is proved.

Corollary 1. *Let (T, X, π) be a transformation group, where X is an arcwise connected Hausdorff space with at least two points and T is compact. Let p be a fixed end point under T. Then there are countably many fixed points.*

Proof. By the definition of end point (e.g., see [**5**]).

Corollary 2. *Let (T, X, π) be a transformation group, where X is an arcwise connected Hausdorff space and T is compact connected. Let p be an end point. Then there are countably many fixed points.*

Proof. Every end point is fixed under a connected group. (See [**3**], the assumption of metric space on p. 124 is not necessary.)

Corollary 3. *Let (T, X, π) be an effective transformation group, where X = either $[a, b]$, a closed interval, or $[a, \infty)$, a closed half-line and T is compact. If a is fixed under T, then T is trivial.*

PROOF. By the theorem, we can show that every orbit contains only one point.

COROLLARY 4. *The only nontrivial compact effective transformation group acting on* $[a, b]$, *a closed interval, is* Z_2, *which is a cyclic group of order* 2.

PROOF. For each $f \in T$, either $f(a) = a$ or $f(a) = b$. Let T_a be the isotropy subgroup of a; then $f^2 \in T_a$ for every $f \in T$. Since T_a is closed, by Corollary 3, T_a = identity. Consequently, f^2 = identity for every $f \in T$. Let f and g be two nontrivial elements in T. Then $fg \in T_a$ and fg = identity or $f = g$. Hence $T = Z_2$. The author benefited from a conversation with Professor A. Shields concerning this result.

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