

# FOURIER COSINE TRANSFORMS WHOSE REAL PARTS ARE NON-NEGATIVE IN A STRIP

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**1. Introduction.** In a recent note [1] we obtained the Poisson integral representation of every function  $u(x, y)$  which is positive and harmonic in a strip,  $-1 < y < 1$ . We here make use of this result to characterize those Fourier cosine transforms

$$\int_0^\infty \cos(x + iy)t \phi(t) dt$$

whose real parts are positive and integrable on  $-\infty < x < \infty$  for each  $y$  in  $-1 < y < 1$ . The characterizing condition on  $\phi(t)$  is that  $\phi(t) \cosh t$  (defined for  $t < 0$  so as to be even) should be real and positive definite. As an example, we have the classical equations

$$\frac{1}{z^2 + 1} = \int_0^\infty \cos zr e^{-r} dr, \quad z = x + iy,$$

$$u(x, y) = \operatorname{Re} \frac{1}{z^2 + 1} = \frac{x^2 - y^2 + 1}{(x^2 - y^2 + 1)^2 + 4x^2y^2} > 0, \quad |y| < 1,$$

$$(1) \quad \int_{-\infty}^\infty u(x, y) dx = \pi, \quad |y| < 1,$$

$$(2) \quad e^{-|r|} \cosh r = \frac{1}{2} \int_{-\infty}^\infty e^{irt} d \left[ U(t) + \frac{1}{\pi} \tan^{-1} \frac{t}{2} \right].$$

Here  $U(t)$  is zero for  $t < 0$  and unity for  $t > 0$ , so that the integrator function in (2) is increasing and bounded. Thus  $e^{-|r|} \cosh r$  is positive definite in confirmation of the theory. Equation (1) can be checked directly or will follow from Corollary 2 below.

An analogous result for the sine-transform is also obtained.

**2. Positive integrable harmonic functions.** In [1] the Poisson integral representation of functions positive and harmonic in a strip was obtained. If such functions are also integrable over the whole doubly-infinite lines of the strip they also have a simple Fourier integral representation, which we now obtain. We use  $H$  and  $L$  to denote the classes of harmonic and integrable (on the whole  $x$ -axis) functions,

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Received by the editors October 6, 1964.

<sup>1</sup> This research was supported by the Air Force Office of Scientific Research.

respectively. Let us state Theorem 4 of [1], altered in notation only. Set

$$(3) \quad Q(x, y) = \frac{1}{4} \frac{\cos \frac{\pi}{2} y}{\cosh \frac{\pi}{2} x + \sin \frac{\pi y}{2}}$$

$$= \frac{1}{\pi} \frac{d}{dx} \tan^{-1} \left[ \frac{\sin \frac{\pi y}{2} + e^{\pi x/2}}{\cos \frac{\pi y}{2}} \right].$$

**THEOREM A.** *A necessary and sufficient condition that  $u(x, y)$  should be non-negative and harmonic in the strip  $-1 < y < 1$  is that*

$$(4) \quad u(x, y) = [Ae^{\pi x/2} + Be^{-\pi x/2}] \sin \frac{\pi y}{2} + \int_{-\infty}^{\infty} Q(x-t, y) d\alpha(t)$$

$$+ \int_{-\infty}^{\infty} Q(x-t, -y) d\beta(t), \quad -1 < y < 1,$$

where  $A \geq 0$ ,  $B \geq 0$ ,  $\alpha(t)$  and  $\beta(t)$  are nondecreasing.

To obtain our basic result we observe that  $Q(x, y)$  is itself a positive definite function of  $x$ . This results from the fact that it is the Fourier transform of a positive function,

$$(5) \quad Q(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt} \frac{\sinh(1-y)t}{\sinh 2t} dt, \quad -1 < y < 1.$$

See p. 36 of [2].

**THEOREM 1.** *A necessary and sufficient condition that  $u(x, y) \in H$ ,  $\in L$ ,  $\geq 0$  for  $-1 < y < 1$  is that*

$$(6) \quad u(x, y) = \int_{-\infty}^{\infty} e^{ixr} \frac{\sinh(1-y)r}{\sinh 2r} g(r) dr + \int_{-\infty}^{\infty} e^{ixr} \frac{\sinh(1+y)r}{\sinh 2r} h(r) dr,$$

where  $g(r)$  and  $h(r)$  are positive definite.

To prove this it will be sufficient to show that under the added condition

$$(7) \quad \int_{-\infty}^{\infty} u(x, y) dx < \infty, \quad -1 < y < 1$$

the representations (4) and (6) are identical. Since every term of (4) is nonnegative it is clear that the above inequality cannot hold unless  $A=B=0$ . From (3) we see that

$$\int_{-\infty}^{\infty} Q(x, y) dx = \frac{1-y}{2}, \quad -1 < y < 1.$$

Hence by Fubini's theorem (7) can then hold if and only if the non-decreasing functions  $\alpha$  and  $\beta$  are also bounded. Then (7) takes the explicit form

$$\int_{-\infty}^{\infty} u(x, y) dx = \frac{1-y}{2} \int_{-\infty}^{\infty} d\alpha(t) + \frac{1+y}{2} \int_{-\infty}^{\infty} d\beta(t) < \infty.$$

Now substituting (5) in (4) we obtain

$$\begin{aligned} u(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha(t) \int_{-\infty}^{\infty} e^{i(x-t)r} \frac{\sinh(1-y)r}{\sinh 2r} dr \\ &\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\beta(t) \int_{-\infty}^{\infty} e^{i(x-t)r} \frac{\sinh(1+y)r}{\sinh 2r} dr. \end{aligned}$$

In view of the boundedness of  $\alpha$  and  $\beta$  we may again apply Fubini's theorem to invert the order of integration, obtaining (6) with

$$\begin{aligned} g(r) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itr} d\alpha(t), \\ h(r) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itr} d\beta(t). \end{aligned}$$

By Bochner's theorem [3]  $g$  and  $h$  are positive definite. This concludes the proof.

**COROLLARY 1.** *If to the conditions of the theorem is added that  $u(x, -y) = u(x, y)$  they become necessary and sufficient that*

$$(8) \quad u(x, y) = \int_{-\infty}^{\infty} e^{ixr} \frac{\cosh yr}{\cosh r} g(r) dr, \quad -1 < y < 1,$$

where  $g(r)$  is positive definite.

For, from (6)

$$u(x, y) = \frac{u(x, y) + u(x, -y)}{2} = \int_{-\infty}^{\infty} e^{ixr} \frac{\cosh yr}{\cosh r} \frac{g(r) + h(r)}{2} dr.$$

The proof is concluded by an obvious change in notation.

A simple change of variable shows that if the strip  $-1 < y < 1$  of Theorem 1 is replaced by  $0 < y < c$ , then (6) becomes

$$(9) \quad u(x, y) = \int_{-\infty}^{\infty} e^{ixr} \frac{\sinh(c-y)r}{\sinh cr} g(r) dr + \int_{-\infty}^{\infty} e^{ixr} \frac{\sinh yr}{\sinh cr} h(r) dr,$$

where  $g$  and  $h$  are positive definite.

**3. The Fourier cosine transform.** The precise statement of the result of §1 follows.

**THEOREM 2.** *The conditions*

A.  $u(x, y) \in H, \geq 0, \in L(-\infty < x < \infty), -1 < y < 1$ ,

B.  $u(-x, y) = u(x, -y) = u(x, y)$

*are necessary and sufficient that*

$$u(x, y) = \operatorname{Re} \int_0^{\infty} \cos(x + iy)t \phi(t) dt, \quad -1 < y < 1,$$

where  $\phi(|t|) \cosh t$  is real and positive definite.

Under Conditions A and B, Corollary 1 shows that

$$u(x, y) = \int_{-\infty}^{\infty} e^{ixr} \frac{\cosh yr}{\cosh r} g(r) dr, \quad -1 < y < 1,$$

for some positive definite function  $g(r)$ . Denote the real and imaginary parts of  $g$  by  $g_1$  and  $g_2$  respectively. Then  $g_1$  is even and positive definite;  $g_2$  is odd. By B,  $u(x, y)$  is even in  $x$ , so that

$$\begin{aligned} u(x, y) &= \int_{-\infty}^{\infty} \cos xr \frac{\cosh yr}{\cosh r} g_1(r) dr \\ (10) \quad &= \operatorname{Re} \int_0^{\infty} \cos(x + iy)r \phi(r) dr, \\ \phi(|r|) &= 2g_1(r)/\cosh r, \quad -\infty < r < \infty. \end{aligned}$$

This proves the sufficiency of Conditions A and B. The necessity follows easily.

**COROLLARY 2.** *Under the Conditions A and B*

$$\int_{-\infty}^{\infty} u(x, y) dx = \pi \phi(0), \quad -1 < y < 1.$$

Write equation (10) as

$$u(x, y) = \int_{-\infty}^{\infty} e^{izr} \frac{\sinh(1-y)r}{\sinh 2r} g_1(r) dr + \int_{-\infty}^{\infty} e^{izr} \frac{\sinh(1+y)r}{\sinh 2r} g_1(r) dr.$$

where

$$g_1(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itr} d\alpha(t).$$

Now apply equation (7) with  $\alpha = \beta$ :

$$\int_{-\infty}^{\infty} u(x, y) dx = \int_{-\infty}^{\infty} d\alpha(t) = 2\pi g_1(0) = \pi \phi(0).$$

In the example of §1,  $\phi(0) = 1$ , so that equation (1) is established.

**4. The Fourier sine-transform.** A companion result to Theorem 2 is the following.

**THEOREM 3.** *The conditions*

A.  $u(x, y) \in H, \in L (-\infty < x < \infty), -1 < y < 1$ ,

B.  $u(x, y) \geq 0, 0 < y < 1$ ,

C.  $u(-x, y) = -u(x, -y) = u(x, y)$

*are necessary and sufficient that*

$$(11) \quad u(x, y) = \text{Im} \int_0^{\infty} \sin(x + iy)t \phi(t) dt, \quad -1 < y < 1,$$

*where  $\phi(t) \sinh t$  is real positive definite ( $\phi$  being odd).*

We first prove the necessity. Assume (11) with

$$\phi(t) \sinh t = g(t) \int_{-\infty}^{\infty} e^{-itr} d\alpha(r),$$

where  $\alpha(r)$  is nondecreasing and bounded. Then

$$(12) \quad u(x, y) = \frac{1}{2} \int_{-\infty}^{\infty} \cos xt \frac{\sinh yt}{\sinh t} g(t) dt$$

$$(13) \quad = \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{\sin \pi y}{\cosh(x-r)\pi + \cos \pi y} d\alpha(r).$$

From (13) we may now verify Conditions A and B; from (12), Condition C.

Conversely, from (9) with  $c=1$ , we have for  $0 < y < 1$

$$(14) \quad u(x, y) = \int_{-\infty}^{\infty} e^{izr} \frac{\sinh(1-y)r}{\sinh r} g(r) dr + \int_{-\infty}^{\infty} e^{izr} \frac{\sinh yr}{\sinh r} h(r) dr,$$

where  $g$  and  $h$  are positive definite. Assuming  $u(x, y)$  odd in  $y$ , we see that  $u(x, 0) = 0$  (A and C). We now show that the first of the integrals (14) is identically zero. From (13) it is clear that it defines a non-negative harmonic function  $v(x, y)$  in  $0 < y < 2$ . The second of the integrals (14) is harmonic in  $-1 < y < 1$  and vanishes for  $y = 0$ . From equation (14),  $v(x, 0) = 0$ . But  $v(x, 1) = 0$ . Hence we may apply the uniqueness result, Corollary 2.2 of [1], to conclude that

$$v(x, y) = [Ae^{\pi x} + Be^{-\pi x}] \sin \pi y, \quad A \geq 0, B \geq 0.$$

Since

$$\int_{-\infty}^{\infty} v(x, y) dx \leq \int_{-\infty}^{\infty} u(x, y) dx < \infty, \quad 0 < y < 1$$

it follows by Condition A that  $A = B = 0$ , so that  $v$  vanishes identically.

Now writing  $h = h_1 + ih_2$  we have

$$\begin{aligned} u(x, y) &= \int_{-\infty}^{\infty} \cos xr \frac{\sinh yr}{\sinh r} h_1(r) dr \\ &= \text{Im} \int_0^{\infty} \sin(x + iy)r \phi(r) dr, \\ \phi(r) &= -\phi(-r) = 2h_1(r)/\sinh r, \quad 0 < r < \infty. \end{aligned}$$

This concludes the proof. As an example we may take

$$\begin{aligned} u(x, y) &= \frac{\pi}{2} \frac{\sin \pi y}{\cosh \pi x + \cos \pi y} = \frac{\pi}{2} \text{Im} \tanh \pi(x + iy) \\ &= \text{Im} \int_0^{\infty} \frac{\sin(x + iy)t}{\sinh t} dt. \end{aligned}$$

Here  $\phi(t) = 1/\sinh t$ , and the function 1 is real and positive definite.

**5. Positive integrable harmonic functions in a half plane.** In [4] we showed that  $u(x, y)$  is harmonic,  $\geq 0$  and integrable in  $x$  ( $-\infty < x < \infty$ ) for  $0 < y < \infty$  if and only if

$$(15) \quad u(x, y) = \int_{-\infty}^{\infty} e^{ixr - y|r|} \psi(r) dr,$$

where  $\psi(r)$  is positive definite. Under these conditions  $u(x, y)$  satisfies the conditions of Theorem 1 in the strip  $0 < y < c$  for every  $c > 0$ . Thus  $u(x, y)$  has the two representations (9) and (15). It is perhaps

useful to record the relations between the functions  $g$ ,  $h$ , and  $\psi$ . By use of the identity

$$e^{-y|r|} = e^{-c|r|} \frac{\sinh yr}{\sinh cr} + \frac{\sinh(c-y)r}{\sinh cr}$$

we see at once that

$$g(r) = \psi(r), \quad h(r) = e^{-c|r|}\psi(r),$$

so that  $g$  is independent of  $c$ , and  $h$  is an exponential multiple of  $g$ . As one would expect the first integral (9) tends to the integral (15) as  $c \rightarrow +\infty$ , the second approaches zero.

#### REFERENCES

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