## REFERENCE

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.


LOUISIANA STATE UNIVERSITY

---

# LOWER BOUNDS FOR SOLUTIONS OF DIFFERENTIAL INEQUALITIES IN HILBERT SPACE

HAJIMU OGAWA[1]

Let $A$ be an operator in a Hilbert space and let $u(t)$ be in the domain of $A$ for each $t \in [0, \infty)$. Assuming $u$ is strongly differentiable, $Au$ strongly continuous and $du/dt$ strongly piecewise continuous, all with respect to $t$, we define

$$Lu = \frac{du}{dt} - Au. \tag{1}$$

In the case where $A$ is symmetric, i.e., $(Au, v) = (u, Av)$, Cohen and Lees [1] obtained lower bounds for solutions of differential inequalities of the form

$$| Lu(t) | \leqq \phi(t) | u(t) |. \tag{2}$$

They proved that if $\phi \in L_p(0, \infty)$ for some $p$ with $1 \leqq p \leqq 2$, then any solution of (2) such that $u(0) \neq 0$ satisfies

$$| u(t) | \geqq Ke^{\lambda t},$$

where $K > 0$ and $\lambda$ are constants depending on the solution. Assuming that $A$ is selfadjoint, Agmon and Nirenberg [2] found a simpler

Presented to the Society, April 13, 1965; received by the editors October 2, 1964.

[1] This work was supported by the Air Force Office of Scientific Research under the contract AFOSR 553-64.

proof of this result, as well as some extensions, by means of convexity theorems. The purpose of this paper is to present still simpler proofs, assuming only that $A$ is symmetric, of the theorem of Cohen and Lees for $p=2$ and of the extensions of Agmon and Nirenberg.

THEOREM. *Suppose $A$ is symmetric and let $u$ be a solution of* (2).

(i) *If $\phi \in L_p(0, \infty)$ for some $p$ with $2 \leqq p \leqq \infty$, then*

$$|u(t)| \geqq |u(0)| \exp[\lambda t - \mu(t+1)^{2-2/p}]. \tag{3}$$

(ii) *If $\phi(t) \leqq K(t+1)^\alpha$, $\alpha > 0$, then*

$$|u(t)| \geqq |u(0)| \exp[\lambda t - \mu(t+1)^{2\alpha+2}]. \tag{4}$$

*In each case, $\lambda$ is a constant depending on $u$, while $\mu$ is a constant depending only on $\phi$.*

PROOF. Assuming $|u(t)| \neq 0$ for all $t \geqq 0$, we first note from (1) and the hypothesis that $A$ is symmetric that

$$\frac{d}{dt} \log |u|^2 = \frac{2(Au, u) + 2 \operatorname{Re}(Lu, u)}{|u|^2} \cdot \tag{5}$$

Moreover, the strong differentiability of $u$ and the strong continuity of $Au$ imply

$$\frac{d}{dt}(Au, u) = 2 \operatorname{Re}\left(Au, \frac{du}{dt}\right) \cdot$$

It follows that

$$\begin{aligned} |u|^4 \frac{d}{dt} \frac{(Au, u)}{|u|^2} &= 2|u|^2 \operatorname{Re}(Au, Au + Lu) \\ &\quad - 2(Au, u) \operatorname{Re}(Au + Lu, u) \\ &= 2|Au + \tfrac{1}{2}Lu|^2 |u|^2 - \tfrac{1}{2}|Lu|^2 |u|^2 \\ &\quad - 2[\operatorname{Re}(Au + \tfrac{1}{2}Lu, u)]^2 + \tfrac{1}{2}[\operatorname{Re}(Lu, u)]^2. \end{aligned}$$

Applying Schwarz's inequality and (2) to this equation, we then find

$$\frac{d}{dt} \frac{(Au, u)}{|u|^2} \geqq -\tfrac{1}{2}\phi^2. \tag{6}$$

For the case (i), we assert that

$$\frac{(Au, u)}{|u|^2} \geqq \lambda - Mt^{1-2/p}, \tag{7}$$

where $\lambda = (Au(0), u(0))/|u(0)|^2$ and $M$ is a constant depending only on $\phi$. For $p=2$ and $p=\infty$, this is an immediate consequence of (6) For $2<p<\infty$, we make use of Hölder's inequality to obtain the estimate

$$\int_0^t \phi^2\,ds \leqq \left(\int_0^t \phi^p\,ds\right)^{2/p}\left(\int_0^t ds\right)^{1-2/p} \leqq Mt^{1-2/p},$$

so (7) follows from the integration of (6).

Applying (7) and (2) to the equation (5), we see that

$$\frac{d}{dt}\log|u(t)| \geqq \lambda - Mt^{1-2/p} - \phi(t).$$

Integrating this inequality and applying Hölder's inequality to the term in $\phi$, we find that

$$\log|u(t)| \geqq \log|u(0)| + \lambda t - \frac{p}{2p-2}Mt^{2-2/p} - Nt^{1-1/p},$$

where $N$ depends only on $\phi$. Since the last two terms are bounded below by

$$-\mu(t+1)^{2-2/p}$$

for some constant $\mu$, (3) follows.

For the case (ii), the lower bound (4) is easily verified if we integrate (6) and apply the resulting inequality, together with (2), to the equation (5).

If $|u(0)|\neq 0$, the assumption that $|u(t)|\neq 0$ for $t>0$ can easily be shown to be valid. For suppose the contrary, and let $t_0$ be the first point where $|u(t)|=0$. Then (3) or (4) holds for $0\leqq t<t_0$, and by continuity the bound also holds at $t_0$, thus contradicting $|u(t_0)|=0$.

## Bibliography

1. P. J. Cohen and M. Lees, *Asymptotic decay of solutions of differential inequalities*, Pacific J. Math. 11 (1961), 1235–1249.

2. S. Agmon and L. Nirenberg, *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math. 16 (1963), 121–239.

University of California, Riverside