# A NOTE ON MEANS OF ENTIRE FUNCTIONS

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Let $f(z)$ denote an entire function of order $\rho$ and lower order $\lambda$, $0 \leqq \lambda, \rho \leqq \infty$, and let us define

$$\sigma_\delta(r) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left| f(re^{i\theta}) \right|^\delta d\theta\right)^{1/\delta}, \quad 0 < \delta < \infty,$$

$$\sigma_{\delta,\kappa}(r) = r^{-\kappa-1}\int_0^r u^\kappa \sigma_\delta(u)\, du, \qquad 0 < \delta < \infty, \quad -1 < \kappa < \infty.$$

For the above two functions, we first prove Theorem 1 below by a method different from that in [4]. Theorem 2 which follows is a simple deduction from Theorem 1.

We require the following lemmas:

LEMMA 1 ([2, THEOREM 148]). *If $\phi$, $\psi$ and $\phi'/\psi'$ are positive increasing functions of $r$ and if $\phi(0) = \psi(0) = 0$, then $\phi/\psi$ is an increasing function for $r > 0$.*

LEMMA 2 ([3, LEMMA 2]).

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log \sigma_{\delta,\kappa}(r)}{\log r} = \begin{matrix}\rho \\ \lambda\end{matrix}, \qquad 0 \leqq \lambda, \qquad \rho \leqq \infty.$$

LEMMA 3. (i) *$r^{\kappa+1}\sigma_\delta(r)$ is a convex function of $r^{\kappa+1}\sigma_{\delta,\kappa}(r)$;*
(ii) *$\sigma_\delta(r)\,/\sigma_{\delta,\kappa}(r)$ is an increasing function of $r$.*

PROOF. Rahman [3, Lemma 3] has proved Lemma 3 (i), with a negligible difference in the definitions of $\sigma_\delta(r)$ and $\sigma_{\delta,\kappa}(r)$ and also assuming $\kappa \geqq 0$ instead of $\kappa > -1$. As in his proof, our definitions easily lead to

$$\frac{d\{r^{\kappa+1}\sigma_\delta(r)\}}{d\{r^{\kappa+1}\sigma_{\delta,\kappa}(r)\}} = \kappa + 1 + \frac{d\{\log \sigma_\delta(r)\}}{d\{\log r\}}.$$

Now, $\log \sigma_\delta(r)$ being known to be a convex function of $\log r$ [1], the right-hand member is an increasing function. Hence the left-hand member too is an increasing function, which proves (i) of Lemma 3.

Lastly, by Lemma 1, (i) implies that $r^{\kappa+1}\sigma_\delta(r)/r^{\kappa+1}\sigma_{\delta,\kappa}(r)$ is an increasing function, i.e., (i) implies (ii).

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THEOREM 1. *For an entire function $f(z)$ of order $\rho$, and lower order $\lambda$, $0 \leqq \lambda$, $\rho \leqq \infty$,*

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\frac{\sigma_\delta(r)}{\sigma_{\delta,\kappa}(r)}\right\}^{1/\log r} = \begin{matrix} e^\rho \\ e^\lambda \end{matrix}.$$

PROOF. It is readily seen from our definitions that

$$\frac{d}{dr}\left[(\kappa+1)\log r + \log \sigma_{\delta,\kappa}(r)\right] = \frac{1}{r}\,\frac{\sigma_\delta(r)}{\sigma_{\delta,\kappa}(r)} \tag{1}$$

so that

$$(\kappa+1)\log\frac{r}{r_0} + \log\sigma_{\delta,\kappa}(r) - \log\sigma_{\delta,\kappa}(r_0) = \int_{r_0}^{r} \frac{\sigma_\delta(u)}{\sigma_{\delta,\kappa}(u)}\cdot\frac{du}{u},$$

or

$$\log\sigma_{\delta,\kappa}(r) = \log\sigma_{\delta,\kappa}(r_0) + \int_{r_0}^{r}\frac{m_{\delta,\kappa}(u)}{u}\,du \tag{2}$$

where

$$m_{\delta,\kappa}(u) = \left[\frac{\sigma_\delta(u)}{\sigma_{\delta,\kappa}(u)} - (\kappa+1)\right] \tag{3}$$

increases as $u$ increases, in virtue of Lemma 3 (ii), and is continuous. Thus for $r > r_0$, (2) gives

$$\log\sigma_{\delta,\kappa}(r) - \log\sigma_{\delta,\kappa}(r_0) < m_{\delta,\kappa}(r)[\log r - \log r_0].$$

Using Lemma 2, we get from this

$$\rho \leqq \limsup_{r\to\infty}\frac{\log m_{\delta,\kappa}(r)}{\log r}, \qquad \lambda \leqq \liminf_{r\to\infty}\frac{\log m_{\delta,\kappa}(r)}{\log r}. \tag{4}$$

Again

$$\log\sigma_{\delta,\kappa}(2r) - \log\sigma_{\delta,\kappa}(r_0) \geqq \int_{r}^{2r}\frac{m_{\delta,\kappa}(u)}{u}\,du \geqq m_{\delta,\kappa}(r)\log 2,$$

which gives

$$\rho \geqq \limsup_{r\to\infty}\frac{\log m_{\delta,\kappa}(r)}{\log r}, \qquad \lambda \geqq \liminf_{r\to\infty}\frac{\log m_{\delta,\kappa}(r)}{\log r}. \tag{5}$$

From (4) and (5) we get

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{\delta,\kappa}(r)}{\log r} = \begin{matrix}\rho\\ \lambda\end{matrix}. \tag{6}$$

The theorem now follows from (3) and (6).

THEOREM 2. *For an entire function $f(z)$ of order $\rho$ and lower order $\lambda$, $0 \leqq \lambda$, $\rho < \infty$,*

$$\log \sigma_{\delta,\kappa}(r) \sim \log \sigma_\delta(r), \qquad r \to \infty.$$

## REFERENCES


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