TRANSLATIONS OF THE IMAGE DOMAINS OF ANALYTIC FUNCTIONS[1]

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If D is a set of complex numbers and a and b are given numbers then by $aD+b$ we mean the set of numbers $ad+b$ where $d\in D$. Translations of D will be denoted by $D+b$, and the number $|b|$ will be called the length of $D+b$.

Let D denote the image of $|z|<1$ under an analytic function $f(z)$. The first theorem proved in this paper is the following: if $f(z)=a_0+z^n+a_{n+1}z^{n+1}+\cdots$ then each translation of D of length less than $\pi/2$ meets D. In the case where $f(z)$ is univalent (and therefore $n=1$) the existence of such a nonzero constant follows from the fact that D covers the circle $|w-a_0|<1/4$.

Next we show that if $f(z)=z^n+a_{n+1}z^{n+1}+\cdots$ then each domain obtained by rotating and translating D meets D, if the translation is of length less than 1. Also, if $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and a translation of D of length δ does not meet D, then $|a_n|\leqq e\delta/\pi$ for $n=1,2,3,\cdots$, and $\sum_{n=1}^{\infty}|a_n|^2\leqq\delta^2$.

LEMMA 1. *If D is the image of $|z|<1$ under an analytic function $f(z)$ and if $D+b\cap D=\varnothing$, then, as n varies over the integers, the sets $\{D+nb\}$ are pairwise disjoint.*

PROOF. A consideration of the functions $f(rz)$, where $0<r<1$, shows that we may assume that $f(z)$ is analytic for $|z|\leqq 1$. The case of constant functions is trivial. Also, it suffices to prove the lemma only in the case where b is real and positive.

If m and n are unequal integers, say $m>n$, then $D+mb\cap D+nb=\varnothing$ is equivalent to $D+lb\cap D=\varnothing$ for some natural number l, namely $l=m-n$. Therefore, an inductive argument establishes the lemma once we prove: for each natural number n, $D+(n+1)b\cap D\neq\varnothing$ implies that either $D+b\cap D\neq\varnothing$ or $D+nb\cap D\neq\varnothing$.

Let n be any natural number and suppose that $w_1\in D+(n+1)b\cap D$. We consider the following curves and points: L_1 and L_2 are the lines of support of D parallel to the real axis with L_1 "below" L_2; w_2 is a boundary point of D on L_1, $w_3=w_2+b$, $w_4=w_2+(n+1)b$; w_5 is a boundary point of $D+b$ on L_2; α is a curve ordered from w_1 to w_2 and lying in D except for w_2; β is a curve ordered from w_4 to w_1 and lying

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in $D+(n+1)b$ except for w_4; γ is a curve lying in $D+b$ except for its endpoints, w_3 and w_5; K is a circle $w=w_3+re^{i\theta}$, $0\leqq\theta\leqq 2\pi$, where $0<r<b$; K_1 is the upper half of K and K_2 is the lower half of K; $w_6=w_3-r$, $w_7=w_3+r$; l_1 is the line segment from w_2 to w_6, l_2 is the line segment from w_7 to w_4; $\delta_1=l_1+K_2+l_2+\beta+\alpha$, $\delta_2=l_1-K_1+l_2+\beta+\alpha$.

Let $n(\delta, w)$ denote the winding number of the closed curve δ with respect to the point w not on δ. Then $n(\delta_2, w_3)=0$, for w_3 is in the unbounded component of the complement of δ_2, and thus $n(\delta_1, w_3)=n(\delta_2, w_3)+n(K, w_3)=n(K, w_3)=1$. Also, $n(\delta_1, w_5)=0$. The curves γ and δ_1 must intersect, otherwise $n(\delta_1, w)$ would be continuous on γ, take on only integral values, and assume at least two different values. This point common to γ and δ_1 must be on $\beta+\alpha$, and it is neither w_2 nor w_4. Therefore, either $D+b\cap D\neq\emptyset$ or $D+b\cap D+(n+1)b\neq\emptyset$. The second possibility is equivalent to $D+nb\cap D\neq\emptyset$.

NOTATION. To each set of complex numbers D we define another set H in the following way. Let E denote the complement of D in the extended plane, and let F be the component of E containing ∞. Let G denote the complement of F in E, and set $H=D\cup G$. G consists of the "holes" in D so that H "fills in" D.

LEMMA 2. *If D is a domain then H is a simply connected domain.*

PROOF. F is closed, since it is a component of the closed set E. Therefore, H, the complement of F, is open.

The lemma is easy to prove if G is void. Otherwise, each component of G, say G_α, has a finite boundary point. Such a point belongs to G_α and is a boundary point of D, and consequently G_α meets the closure of D. Therefore, since D and G_α are connected, $H_\alpha=D\cup G_\alpha$ is connected. Then H is connected, for $H=\bigcup_\alpha H_\alpha$, and the sets $\{H_\alpha\}$ are connected and pairwise have common points. This completes the proof that H is a domain. Also, H is simply connected since F is connected in the extended plane.

LEMMA 3. *Let D be the image of $|z|<1$ under an analytic function $f(z)$, and let $D'=aD+b$ and $H'=aH+b$, where $|a|=1$. If $D\cap D'=\emptyset$ then $H\cap H'=\emptyset$.*

PROOF. We assume that G is nonvoid, otherwise the lemma is trivial. This also takes care of constant functions so that henceforth D is a domain.

A consideration of the functions $f(rz)$, where $0<r<1$, shows that we may assume that $f(z)$ is analytic for $|z|\leqq 1$.

First we shall show that there is a point in D that is not in G', where $G' = aG + b$. Let $d(A)$ denote the diameter of the set A. Suppose w_1 and w_2 are points on the boundary of G satisfying $d(G) = |w_1 - w_2|$, and let L be the ray beginning at w_1, that, if extended to a full line, would pass through w_2. There are points on L belonging to F, for D is bounded and w_1 is the only point both on L and in G. Therefore, since F and G are separated there is a point w_3 that lies on L, is a boundary point of F, and is different from w_1. Thus, $d(G) < |w_3 - w_2|$. Also, $|w_3 - w_2| \leqq d(D)$, for w_2 and w_3 are boundary points of D. This proves that $d(G) < d(D)$. But $d(G) = d(G')$ since $|a| = 1$, and thus $d(G') < d(D)$. This implies that there is at least one point in D that is not in G'.

We will show that $D \cap G' = \emptyset$. On the contrary, suppose that $w_1 \in D \cap G'$. Let w_2 be any point such that $w_2 \in D$ and $w_2 \notin G'$, and let α be a curve lying in D with the endpoints w_1 and w_2. Since $w_1 \in G'$ and $w_2 \notin G'$ there is a point w_3 such that w_3 is on α and w_3 is a boundary point of G'. Thus, w_3 is a boundary point of D' and $w_3 \in D$. Because of the openness of D this contradicts $D \cap D' \neq \emptyset$.

The roles of D and D' are interchangeable. In particular, the conditions $D' = aD + b$ and $|a| = 1$ are equivalent to the conditions $D = a'D' + b'$ and $|a'| = 1$. Thus, by interchanging D and D' and G and G' in our previous argument we obtain $D' \cap G = \emptyset$.

The lemma is proven once we show that $G \cap G' = \emptyset$. On the contrary, suppose that $G \cap G' \neq \emptyset$. Then $G \cap G'$ has a boundary point w_1. Because of the symmetry in the roles of G and G' we may assume that w_1 is a boundary point of G. Then w_1 is also a boundary point of D. Let w_2 be any point such that $w_2 \in D$ and $w_2 \notin G'$, and let α be a curve having endpoints w_1 and w_2 and lying in D except for w_1. Let G'' be the component of G' containing w_1, and set $E' = aE + b$, where E is defined in the Notation previous to Lemma 2. If α does not intersect D' then $G'' \cup \alpha$ is a connected subset of E', and this violates the maximality of the component G'', for $w_2 \in \alpha$ and $w_2 \notin G''$. Therefore, there is a point that lies on α and is in D', and it is not w_1. This contradicts $D \cap D' = \emptyset$.

THEOREM 1. *Let D be the image of $|z| < 1$ under an analytic function $f(z) = a_0 + z^n + a_{n+1}z^{n+1} + \cdots$. Each translation of D of length less than $\pi/2$ meets D.*

PROOF. Suppose that $D + b \cap D = \emptyset$. With $a = 1$ Lemma 3 implies that $H + b \cap H = \emptyset$, and, in particular, H is not the whole plane. According to Lemma 2, H is a simply connected domain. Therefore, there is a function $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ analytic and univalent for $|z| < 1$ and mapping $|z| < 1$ onto H so that $g(0) = f(0)$. Then $f(z)$

is subordinate to $g(z)$, and consequently there exists a function $\phi(z) = c_1z+c_2z^2+\cdots$ that is analytic and satisfies $|\phi(z)|<1$ for $|z|<1$ and such that $f(z)=g(\phi(z))$. Using the form of $f(z)$, $f(z)=g(\phi(z))$ and $b_1\neq 0$ one obtains successively $c_1=0$, $c_2=0, \cdots, c_{n-1}=0$, $c_nb_1=1$. The inequality $|c_n|\leqq 1$ implies that $|b_1|\geqq 1$.

Since $H+b\cap H=\varnothing$ Lemma 1 may be applied to H, and this implies that the function $h(z)=(2\pi i/b)g(z)$ assumes no pair of values differing by an integral multiple of $2\pi i$. Together with the fact that $h(z)$ is univalent for $|z|<1$, this shows that the function $k(z)=e^{h(z)}$ is univalent for $|z|<1$. Also, the function $l(z)=(k(z)-k(0))/k'(0) = z+d_2z^2+\cdots$ is analytic and univalent for $|z|<1$, and $l(z) \neq -k(0)/k'(0)$ because $k(z)\neq 0$. Since $k(0)/k'(0)=b/2\pi ib_1$ an application of the 1/4-theorem to $l(z)$ implies that $|b/2\pi ib_1|\geqq 1/4$, $|b|\geqq(\pi/2)|b_1|$. Using $|b_1|\geqq 1$ we obtain $|b|\geqq \pi/2$, and this proves the theorem.

Knowing the functions for which the 1/4-theorem is precise and those for which $|c_n|\leqq 1$ is precise one can show that this theorem is exact only for the functions $f(z)=(\bar{\epsilon}/2)\log[(1+\epsilon z^n)/(1-\epsilon z^n)]$ where $|\epsilon|=1$. These functions map $|z|<1$ onto a strip of width $\pi/2$.

THEOREM 2. *Let D be the image of $|z|<1$ under an analytic function $f(z)=z^n+a_{n+1}z^{n+1}+\cdots$. Each domain obtained by rotating and translating D meets D, if the translation is of length less than* 1.

PROOF. Suppose that $aD+b\cap D=\varnothing$ where $|a|=1$. Because of Lemmas 2 and 3 the argument given in the proof of Theorem 1 shows that there exists a function $g(z)=b_1z+b_2z^2+\cdots$ analytic and univalent for $|z|<1$ and mapping $|z|<1$ onto H such that $|b_1|\geqq 1$, and $aH+b\cap H=\varnothing$.

The functions $g(z)$ and $h(z)=ag(z)+b$ are univalent and have no values in common. The same properties are held by the functions $G(z)=1/g(z)=A_{-1}/z+\sum_{n=1}^{\infty}A_nz^n$ and $H(z)=1/h(z)=\sum_{n=0}^{\infty}B_nz^n$, and therefore $\sum_{n=0}^{\infty}n(|A_n|^2+|B_n|^2)\leqq|A_{-1}|^2$ [4, p. 226, problem 14]. In particular, $|B_1|\leqq|A_{-1}|$, and from this we obtain $|b|\geqq|b_1|$, since $B_1=-ab_1/b^2$, $A_{-1}=1/b_1$ and $|a|=1$. Because $|b_1|\geqq 1$ this shows that $|b|\geqq 1$, and this proves the theorem.

It is not difficult to show that this theorem is sharp only for the functions $f(z)=z^n/(1+\epsilon z^n)$ where $|\epsilon|=1$. Each such function is extremal, since it maps $|z|<1$ onto a half plane bounded by a line that is tangent to the circle $|w|=1/2$.

LEMMA 4. *Let D be the image of $|z|<1$ under an analytic function, and suppose that a translation of D of length δ does not meet D. Then* *each line parallel to the direction of translation intersects D in a set whose linear measure does not exceed* δ.

PROOF. It suffices to consider the case where $D+b\cap D=\emptyset$, b is real and positive and the line in question is the x-axis. Let A denote the intersection of D with the x-axis and let $m(S)$ denote the measure of the set of real numbers S. Except for the trivial cases where the function is constant or A is void, A is an open set of real numbers, and therefore $A=\bigcup_{n=1}^{\infty} I_n$ where each I_n is an open interval. $D+b\cap D=\emptyset$ implies that $m(I_n)\leqq b$ for each n. If I_n does not contain an integral multiple of b then there is an integer n' such that the set $I_n' = I_n+n'b$ lies in the interval $0\leqq x\leqq b$. If I_n contains a multiple of b, say $n'b$, then the point $n'b$ breaks up I_n into two intervals, J_1 and J_2, so that $J_1' = J_1-(n'-1)b$ and $J_2' = J_2-n'b$ lie in the interval $0\leqq x\leqq b$. In this case let $I_n' = J_1'\cup J_2'$. Then, for all n, I_n' lies in the interval $0\leqq x\leqq b$, and $m(I_n')=m(I_n)$. According to Lemma 1 the sets $\{I_n'\}$ are pairwise disjoint, and therefore $m(A)=\sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} m(I_n')\leqq b$.

THEOREM 3. *Let D be the image of* $|z|<1$ *under an analytic function* $f(z)=\sum_{n=0}^{\infty} a_n z^n$. *If a translation of D of length* δ *does not meet D, then*

$$|a_n| \leqq (e/\pi)\delta, \quad \text{for } n\geqq 1, \tag{1}$$

$$\sum_{n=1}^{\infty} |a_n|^2 \leqq \delta^2. \tag{2}$$

PROOF. Suppose that $D+b\cap D=\emptyset$ and $|b|=\delta$. Let z be any complex number such that $|z|<1$ and $f'(z)\neq 0$, and set $F(w) = f((w+z)/(1+\bar{z}w)) = A_0+A_1w+\cdots$ so that $A_1=f'(z)(1-|z|^2)$. The function $G(w)=F(w)/A_1$ maps $|w|<1$ onto the domain $D^* = D/A_1$ so that $D^*+(b/A_1)\cap D^*=\emptyset$. Applying Theorem 1 (with $n=1$) to $G(w)$ we obtain $|b/A_1|\geqq \pi/2$, and therefore

$$|f'(z)| \leqq \frac{2}{\pi}\,\frac{\delta}{1-|z|^2}. \tag{3}$$

This estimate also holds if $f'(z)=0$.

Using (3) and the formula

$$na_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f'(z)}{z^n}\,dz$$

we obtain $n|a_n|\leqq (2/\pi)\delta/(1-r^2)r^{(n-1)}$, valid for $n\geqq 1$ and for each r satisfying $0<r<1$. For $n>1$ the choice of r such that $r^2=(n-1)\cdot(n+1)^{-1}$ gives the estimate

$$|a_n| \leqq \frac{\delta}{\pi}\,\frac{n+1}{n}\left(1+\frac{2}{n-1}\right)^{(n-1)/2}.$$

The function of n on the right of this inequality increases for $n>1$ and approaches $\delta e/\pi$ as $n\to\infty$. Therefore, $|a_n|<(e/\pi)\delta$ for $n\geqq 2$. For $n=1$ we have the sharp estimate $|a_1|\leqq(2/\pi)\delta$.

To prove (2) we may assume that $f(z)$ is not a constant. Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be a function analytic and univalent for $|z|<1$ and mapping $|z|<1$ onto H so that $g(0)=f(0)$. Since $f(z)$ is subordinate to $g(z)$ for $|z|<1$ it follows that

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leqq \sum_{n=1}^{\infty} |b_n|^2 r^{2n} \tag{4}$$

for each r satisfying $0<r<1$ [3, p. 484, Theorem 2 with $k=2$].

Lemma 3 (with $a=1$) implies that $H+b\cap H=\emptyset$ so that the estimate (3) also applies to $g'(z)$. Therefore, if $z=re^{i\theta}$ then

$$\begin{aligned} |g(z)-g(0)| &= \left|\int_0^z g'(w)dw\right| \leqq \int_0^r |g'(e^{i\theta}t)|\,dt \\ &\leqq \int_0^r \frac{2}{\pi}\,\frac{\delta}{1-t^2}\,dt = \frac{\delta}{\pi}\log\frac{1+r}{1-r}. \end{aligned}$$

Thus, $g(z)$ maps $|z|\leqq\rho$ onto a set S contained in a circle of radius $R=(\delta/\pi)\log[(1+\rho)/(1-\rho)]$. Together with Lemma 4, this implies that the area of S satisfies $A\leqq 2\delta R$ and this inequality is equivalent to

$$\pi\sum_{n=1}^{\infty} n|b_n|^2\rho^{2n} \leqq \frac{2\delta^2}{\pi}\log\frac{1+\rho}{1-\rho}. \tag{5}$$

Expressing the right side of (5) in a series in ρ, dividing both sides by ρ and integrating from 0 to r, we obtain

$$\sum_{n=1}^{\infty} |b_n|^2 r^{2n} \leqq \frac{8\delta^2}{\pi^2}\sum_{n=0}^{\infty}\frac{r^{2n+1}}{(2n+1)^2}. \tag{6}$$

Because of (4) this shows that

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leqq \frac{8\delta^2}{\pi^2}\sum_{n=0}^{\infty}\frac{r^{2n+1}}{(2n+1)^2}. \tag{7}$$

Using $r<1$ and $\sum_{n=0}^{\infty} 1/(2n+1)^2=\pi^2/8$ we obtain

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leqq \delta^2, \tag{8}$$

and (2) follows from (8) by letting $r \to 1$.

REMARKS. 1. Theorem 1 can be proved using either Lemma 1 or Lemma 4 and results obtained by the principle of symmetrization (see [2, Theorems 4.10 and 4.15]).

2. Theorem 3 should be compared with the following: if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic for $|z| < 1$, and if the values of $f(z)$ lie in a strip of width δ, then $|a_n| \leqq (2/\pi)\delta$ for $n \geqq 1$, and $\sum_{n=1}^{\infty} |a_n|^2 \leqq \delta^2/2$ (see: [5, p. 130, problem 238], [6, Theorem 10; 7]).

3. In [1] (see Theorem 3), the following is proved: if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic for $|z| < 1$, and if the image domain D does not contain arbitrarily large circles, then the sequence $\{a_n\}$ is bounded. This can be improved to $a_n \to 0$ as $n \to \infty$ in the case where H does not contain arbitrarily large circles. This is an immediate consequence of the convergence of $\sum_{n=0}^{\infty} |a_n|^2$, and this can be proven using an argument like that given in Theorem 3 with the fact that $|g'(z)| = O((1-|z|)^{-1})$ [1, p. 431, (3.3)], where $g(z)$ is the function in Theorem 3.

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