THREE RESULTS FOR LOCALLY COMPACT GROUPS CONNECTED WITH THE HAAR MEASURE DENSITY THEOREM

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In [6] I have given a generalization of the Lebesgue measure density theorem to the Haar measure on a locally compact group. A slight modification of the argument given there yields the following:

THEOREM A. *Let A be a σ-bounded subset and μ a left-invariant Haar measure of a locally compact group G. Then there exists a sequence U_n of bounded measurable neighborhoods of the unit e of G such that $\mu^*(A \cap U_n x)/\mu(U_n x) \to 1$, $n \to \infty$ for almost all x out of a measurable cover of A. The sequence U_n may be chosen out of a given basis of bounded measurable neighborhoods and simultaneously for a countable family of sets A.*

1. A. Beck, H. H. Corson and A. B. Simon [1] have considered the inner points of a product AB of two subsets of a locally compact group. They prove that $\operatorname{int}(AB)$ is not empty whenever $\mu(A) > 0$, $\mu^*(B) > 0$. The following theorem shows that $\operatorname{int}(AB)$ is quite large.

THEOREM 1. *Let A, B be σ-bounded subsets of a locally compact group G, A measurable and C a measurable cover of B. Then there exist subsets A^*, C^* of A, C equal in measure such that $A^*C^* \subset \operatorname{int}(AB)$.*

PROOF. By Theorem A we choose a sequence U_n of compact neighborhoods of e for A^{-1} and B. We get

$$\frac{\mu(A^{-1}a \cap U_n)}{\mu(U_n)} = \frac{\mu(A^{-1} \cap U_n a^{-1})}{\mu(U_n a^{-1})} \to 1, \qquad \frac{\mu^*(Bb^{-1} \cap U_n)}{\mu(U_n)} \to 1$$

for almost all $a \in A$, $b \in C$. Let A^*, C^* be the sets of those $a \in A$, $b \in C$ for which these limits are actually 1. Suppose $a \in A^*$, $b \in C^*$ and $ab \notin \operatorname{int}(AB)$. For every neighborhood V of e there exists $x \in V$, $axb \notin AB$. Consequently $A^{-1}ax \cap Bb^{-1} = \emptyset$ and

$$\begin{aligned} \mu^*(U_n V) &\geqq \mu(A^{-1}ax \cap U_n V) + \mu^*(Bb^{-1} \cap U_n V) \\ &\geqq \mu(A^{-1}ax \cap U_n x) + \mu^*(Bb^{-1} \cap U_n) \\ &= \mu(A^{-1}a \cap U_n)\Delta(x) + \mu^*(Bb^{-1} \cap U_n). \end{aligned}$$

If V is a small enough neighborhood of e, it follows that $\Delta(x)$ is arbitrarily close to 1, and that $\mu^*(U_n V)$ is arbitrarily close to $\mu(U_n)$

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for any fixed n since each U_n is compact and μ is regular. Thus, for each n,

$$\mu(U_n) \geqq \mu(A^{-1}a \cap U_n) + \mu^*(Bb^{-1} \cap U_n).$$

However, for n large enough, both

$$\frac{\mu(A^{-1}a \cap U_n)}{\mu(U_n)} \quad \text{and} \quad \frac{\mu^*(Bb^{-1} \cap U_n)}{\mu(U_n)}$$

are greater than 1/2, yielding the contradiction

$$\mu(U_n) \geqq \mu(A^{-1}a \cap U_n) + \mu^*(Bb^{-1} \cap U_n) > \mu(U_n).$$

REMARK. The theorem yields explicit estimates of the measure of $\text{int}(AB)$. For $\mu(A) > 0$ and $a \in A^*$ we have

$$\mu(\text{int}(AB)) \geqq \mu^*(A^*C^*) \geqq \mu(aC^*) = \mu(C^*) = \mu^*(B).$$

Other estimates for $\mu(AB)$ (e.g. M. Kneser [**4**]) can be used in the same manner.

2. On the basis of Theorem A I have studied *e-joint semigroups* in locally compact groups [**7**]; these are semigroups N such that $\mu_*(N \cap U) > 0$ for all neighborhoods U of e. They have the fundamental property that their boundary $\overline{N} \setminus \text{int}(N)$ is of measure zero.

M. Zorn [**2**, p. 157] has proved the following:

THEOREM B. *If N is a semigroup in a topological group which is of the second category at e and satisfies the condition of Baire, then* $\text{int}(N) = \text{int}(\overline{N})$ *and* $\text{int}(N)$ *is dense in N.*

THEOREM 2. *A semigroup N in a locally compact group G is e-joint if and only if it is of the second category at e and satisfies the condition of Baire.*

PROOF. In a locally compact group G every nonempty open set A is of the second category. For suppose $A = \bigcup_{i=1}^{\infty} A_i$, A_i nowhere dense in G. For each i and each nonempty open set U there exists a nonempty bounded open set V with $\overline{V} \subset U$, $\overline{V} \cap A_i = \emptyset$. Starting with $V_0 = A$ we construct a sequence of nonempty bounded open sets V_n, $n \geqq 1$ with $\overline{V}_n \subset V_{n-1}$, $\overline{V}_n \cap A_n = \emptyset$. They form a filter-basis on the compact set $\overline{V}_1$ having a cluster point $x \in \overline{V}_1 \subset V_0 = A$, $x \in \overline{V}_n$, $n \geqq 1$. This implies the contradiction $x \notin A_n$, $n \geqq 1$ and $x \notin A$.

If N is e-joint, $\mu(\text{int}(N) \cap U) = \mu(N \cap U) > 0$ for all open neighborhoods U of e; hence $\text{int}(N) \cap U$ is of the second category, and so is N at e. Further, as the measure of the boundary of N is zero, the boundary has empty interior; and that implies the condition of Baire.

Conversely, if the conditions of Theorem B hold for N, we have $e\in\overline{N}=(\mathrm{int}(N))^-$, therefore $\mathrm{int}(N)\cap U\neq\emptyset$ for all open neighborhoods U of e and $\mu_*(N\cap U)\geqq\mu(\mathrm{int}(N)\cap U)>0$, so N is e-joint.

3. A *subadditive function* f on a locally compact group G is a function whose values are real or $\pm\infty$ and which satisfies $f(xy)\leqq f(x)+f(y)$.[1] Such functions have been studied by E. Hille and M. Zorn [3], [2], R. A. Rosenbaum [8] and others, mainly in connection with semigroups.

With a subadditive function f there are associated the functions $f_*(x)=\liminf_{y\to x}f(y)$ and $f^*(x)=\limsup_{y\to x}f(y)$ which are subadditive, lower resp. upper semicontinuous and satisfy $f_*\leqq f\leqq f^*$.

THEOREM 3. *Let f be a subadditive function on a locally compact group G satisfying* $\liminf_{x\to e}f^*(x)\leqq 0$. *Then f is continuous almost everywhere.*

PROOF. Let R be the additive group of the real numbers and $N\subset G\times R$ the set $\{(x,\alpha)\mid f(x)\leqq\alpha\}$. N is a semigroup, $\mathrm{int}(N)=\{(x,\alpha)\mid f^*(x)<\alpha\}$ and $\overline{N}=\{(x,\alpha)\mid f_*(x)\leqq\alpha\}$. The condition $\liminf_{x\to e}f^*(x)\leqq 0$ implies (and is actually equivalent to) the e-jointness of N. Hence the boundary of N is of measure zero, and this measure is equal to $\int_G(f^*(x)-f_*(x))d\mu$. Consequently $f_*(x)=f^*(x)$ almost everywhere.

REMARKS. If $\liminf_{x\to e}f^*(x)>0$, f may be discontinuous everywhere.

$\liminf_{x\to e}f^*(x)\leqq 0$ implies $=0$ or $=-\infty$. If it is $=-\infty$, f may be described as follows: There exists an e-joint semigroup N in G (whose boundary is of measure zero) such that $f(\mathrm{int}(N))=-\infty$, $f(\mathrm{int}(N'))=+\infty$.

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[1] With $(+\infty)+(-\infty)=+\infty$.