A REMARK ON THE EXISTENCE OF A G-STRUCTURE

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The purpose of this note is to show that a 0-deformable tensor field [4] defines a G-structure, i.e., a subbundle of the frame bundle.[2] We shall prove this statement in a slightly more general form. I am indebted to Professor K. Nomizu for calling my attention to this problem [Math. Reviews 27 (1964) #678], and to Professor T. Tamagawa for the valuable suggestion he made for the proof.

In [3, p. 294, Theorem], Crittenden shows that a cross section X of an associated bundle (W, G, F, M, ρ) of a principal bundle (P, G, M, π) is parallelizable if and only if $f_X(P) = \beta_G f$, where $\beta_G f$ is an orbit by the action $\beta: G \times F \to F$, through some fixed $f \in F$. Here $f_X: P \to F$ is the differentiable map defined by $f_X(p) = F(p)^{-1}(X(\pi(p)))$ for $p \in P$, where $F(p): F \to F(\pi(p))$ is the map induced from $P \times F \to W$.

He further asserts that if X is parallelizable, $B = f_X^{-1}(f)$ is a bundle with group $K = \{g \in G \mid \beta_g f = f\}$. In the proof of this assertion, the key idea is that $f'_X: P \to G/K$ determines a cross section in the associated bundle with fibre G/K, where f'_X is defined by $f_X = \iota \circ f'_X$, ι being the map $G/K \to \beta_G f \subset F$ induced from $g \to \beta_g f$. However this f'_X is not necessarily differentiable.

This is precisely the point that Bernard worries about in the more special setting, where W is a tensor bundle (thus F is a vector space and G is the general linear group $GL(n, R)$), in [1, p. 211, Proposition III.2]. For f'_X to be differentiable, it suffices that f'_X be continuous, and for the latter it suffices that $(\iota, G/K)$ be a regular submanifold of F.

Let us recall that "if G is a locally compact topological group which is a countable union of compact sets, S is a locally compact space, and if G acts on S as a transitive group of transformations then G/H_p is homeomorphic to S, where H_p is the isotropy subgroup of G at a point p of S." The proof of this can be obtained from the proof in Pontrjagin [5, Theorem 12].

This shows, that, in order that $(\iota, G/K)$ be a regular submanifold of F, it suffices that $\beta_G f$ be locally compact. In the rest we shall show the following lemma:

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[2] This has been proved by Y. C. Wong [7, pp. 73–75]. We present a different proof.

LEMMA. *If F is the real vector space R^n, and G a real algebraic group contained in $GL(n, R)$, then $\beta_G f$ is locally compact.*

COROLLARY. *Under the condition of the lemma, if X is parallelizable then $B = f_X^{-1}(f)$ is a subbundle with group K of the principal bundle P.*

PROOF OF LEMMA.[3] By a real algebraic group G contained in $GL(n, R)$ we mean a group consisting of all invertible real $n \times n$ matrices whose coefficients annihilate some set of polynomials with real coefficients in n^2 indeterminates. G is acting on R^n.

Let $x_0 \in R^n$ be fixed and consider the orbit $G \cdot x_0$. If G is irreducible (as an algebraic set) then $G \cdot x_0$ is also irreducible. If G is not irreducible, let the finite number of irreducible components of G be denoted by G_i. If $G_i \cdot x_0 \cap G_j \cdot x_0 \neq \emptyset$, then $G_i \cdot x_0 = G_j \cdot x_0$. Hence, in order to prove that $G \cdot x_0$ is locally compact (in the induced topology from the ordinary euclidean topology on R^n), it suffices to assume G to be irreducible.

Now assuming G to be irreducible, let V be the smallest algebraic set in R^n containing $G \cdot x_0$. V is irreducible. From [2, p. 191, Lemma 2, and p. 180, Proposition 13], we see that all the points of $G \cdot x_0$ are simple points of V. By Whitney [6], we know that $V = M_1 \cup V_1$, where $V_1 = V - M_1$, V_1 is void or a proper algebraic set in V, and M_1 is a manifold consisting of all simple points of V.

Hence $G \cdot x_0 \subset M_1$. $G \cdot x_0$ is an open submanifold of M_1 (where we are considering the topology on M_1 induced from the ordinary euclidean topology of R^n). Hence as $G \cdot x_0$ is an open set of M_1, which in turn is an open set of V, which in turn is a closed set of R^n with the euclidean topology, we conclude that $G \cdot x_0$ is locally compact. Q.E.D.

REFERENCES

1. D. Bernard, *Sur la géométrie différentielle des G-structures*, Ann. Inst. Fourier (Grenoble) **10** (1960), 157–269.

2. C. Chevalley, *Théorie des groupes de Lie*. III, Hermann, Paris, 1955.

3. R. Crittenden, *Covariant differentiation*, Quart. J. Math. Oxford (2) **13** (1962), 285–298.

4. E. T. Kobayashi, *A remark on the Nijenhuis tensor*, Pacific J. Math. **12** (1962), 963–977.

5. L. S. Pontrjagin, *Topologische Gruppen*, Teubner, Leipzig, 1957–1958.

6. H. Whitney, *Elementary structure of real algebraic varieties*, Ann. of Math. (2) **66** (1957), 545–556.

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[3] The idea of this proof was taken from the proof of Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962), 485–535, p. 495, Proposition 2.3.

7. Y. C. Wong, *Existence of linear connections with respect to which given tensor fields are parallel or recurrent*, Nagoya Math. J. 24 (1964), 67–108.

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A NOTE ON A REDUCIBLE CONTINUUM

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In [4], Knaster shows that there exists an irreducible compact metric continuum M which has a monotone continuous decomposition G such that each element of G is nondegenerate and M/G is an arc. Also, he raised the question as to whether there existed an irreducible continuum M which has a monotone continuous decomposition G such that each element of G is an arc and M/G is an arc. E. E. Moise settled this question in the negative in [5]. In [3], M. E. Hamstrom showed that if G is a monotone continuous decomposition of a compact metric continuum such that each element of G is a nondegenerate continuous curve and M/G is an arc, then it is not the case that M is irreducible. E. Dyer generalized this result by showing in [2] that if M is a compact metric continuum and G is a monotone continuous decomposition of M such that each element of G is nondegenerate and decomposable, then it is not the case that M is irreducible. A purpose of this note is to extend Dyer's result somewhat.

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THEOREM 1. *Let M denote a compact metric continuum and G a nondegenerate monotone continuous decomposition of M each of whose elements is nondegenerate. If H is a subcollection of G each of whose elements is snakelike and indecomposable, and if H^* is dense in M, then uncountably many elements of G are indecomposable.*

PROOF. Let I_1 denote an element of H, and let C_1 denote the first chain in a sequence of defining chains for I_1, and let L_1 and L_2 denote the end links of C_1. Since H^* is dense in M, and G is a continuous collection, C_1 contains two elements $I(10)$ and $I(11)$ of H such that $I(10)$ and $I(11)$ intersects every link of C_1. Let $\{C_n(10)\}$ and $\{C_n(11)\}$ denote chain sequences which define $I(10)$ and $I(11)$ respectively.

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