ON THE DIOPHANTINE EQUATION $x^3+y^3+z^3=x+y+z$

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1. The remarks in this note on the Diophantine equation

$$x^3 + y^3 + z^3 = x + y + z \tag{1}$$

are prompted by Edgar's recent note [1]. In order "to avoid certain trivial solutions" he assumes that $x \geqq y \geqq 0$, $z<0$ and $x \neq -y$. Using a method of S. D. Chowla and others (a reference not accessible to me) he obtains infinitely many solutions of (1) subject to the further conditions

$$x + y + z = m, \qquad x + y = km, \qquad x + z \neq 0 \tag{2}$$

in each of the following cases (i) $k=3$ (Chowla), (ii) $k=12$ (Edgar), (iii) $k=16/3$ (Edgar).

In this note I show that each of the trivial solutions $(h, 1, -h)$ where $|h| \geqq 2$, gives rise to infinitely many nontrivial solutions and that nontrivial solutions likewise generate others.

As an example the equation

$$N^2 - 85M^2 = -4 \tag{3}$$

has infinitely many integral solutions (N, M), both odd or both even. The integers

$$x = \tfrac{1}{2}(M + N), \qquad y = \tfrac{1}{2}(M - N), \qquad z = -4M \tag{4}$$

will always satisfy (1). And for those solutions

$$x + y + z = -3M, \qquad x + y = M.$$

These solutions were obtained in fact by the method below from the nontrivial solution $(5, -4, -4)$.

The equation

$$3N^2 - 31M^2 = -4 \tag{5}$$

also has infinitely many solutions with M, N of like parity: the equations

$$x + y = -M, \qquad x - y = N, \qquad z = 2M$$

will yield solutions of (1). Equation (5) was derived from the trivial solution $(-2, 1, 2)$ of (1).

The equation

Received by the editors May 19, 1965.

$$(6) \qquad 5N^2 - 62M^2 = 2$$

has infinitely many solutions in integers N(even) and M: the smallest solution appears to be (412, 117). Determine x, y, z by the equations

$$x + y = 10M, \qquad x - y = N, \qquad z = -7M:$$

then x, y, z are integers which satisfy (1). One solution is therefore $x=791$, $y=379$, $z=-819$.

2. Suppose that (x, y, z) is a solution of (1). Any permutation yields another solution (not necessarily distinct). Also $(-x, -y, -z)$ is a solution. In general 12 solutions arise from a given solution.

Suppose that $x+y$ and z are not both zero. Define integers m, n, a, c uniquely by the following equations

$$(7) \quad x + y = am, \quad z = -cm, \quad x - y = n, \quad (a, c) = 1, \quad m \geqq 1.$$

Then from the identity

$$4(x^3 + y^3 + z^3 - x - y - z) = m\{3an^2 + (a^3 - 4c^3)m^2 - 4(a - c)\}$$

we see that the integers (m, n) $(m \geqq 1)$ satisfy the Diophantine equation

$$(8) \qquad (a^3 - 4c^3)M^2 + 3aN^2 = 4(a - c).$$

Conversely, suppose that integers a and c exist such that (8) is solvable in integers $M \neq 0$, N with aM, N of like parity; then the equations

$$X + Y = aM, \qquad Z = -cM, \qquad X - Y = N$$

give integers X, Y, Z which satisfy (1).

If in addition the integer D defined by

$$(9) \qquad D = 3a(4c^3 - a^3)$$

is positive and not a square, then the equation (8), having one solution (M, N) with aM, N of same parity, will have infinitely many such solutions, by a classical theorem on indefinite binary quadratic forms.

As an example, take the trivial solution

$$(x, y, z) = (h, 1, -h)$$

where h is an integer, $|h| \geqq 2$. Equation (8) becomes

$$(10) \qquad \begin{gathered} 3(h + 1)N^2 - (3h^3 - 3h^2 - 3h - 1)M^2 = 4, \\ D = 3(h + 1)(3h^3 - 3h^2 - 3h - 1). \end{gathered}$$

It is easily seen that $D>0$ and that D is not a square whenever $|h| \geqq 2$. Now (10) has the solution $M=1$, $N=h-1$: it has therefore infinitely many solutions such that $(h+1)$ M, N have the same parity. I omit the proof.

Here are a few examples of (10):

$$\begin{aligned} 9N^2 - 5M^2 &= 4, & 31M^2 - 3N^2 &= 4,\\ 3N^2 - 11M^2 &= 1, & 100M^2 - 6N^2 &= 4,\\ 15N^2 - 131M^2 &= 4, & 229M^2 - 9N^2 &= 4,\\ 18N^2 - 284M^2 &= 4, & 109M^2 - 3N^2 &= 1. \end{aligned}$$

3. Each solution (x, y, z) of (1) gives rise in general to three pairs of integers (a, c) and hence to three binary forms. Suppose (x_1, y_1, z_1) and (x_2, y_2, z_2) are two solutions of (1) derived from two pairs (N_1, M_1), (N_2, M_2), belonging to a particular binary form corresponding to a pair (a, c): suppose also that $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$ or to $(-x_2, -y_2, -z_2)$. Then the two remaining binary forms deducible from the triad (x_1, y_1, z_1) are distinct from those deducible from the triad (x_2, y_2, z_2). In this way further sets of solutions can be generated.

As an example the trivial solution $(2, 1, -2)$ leads to the form $9N^2-5M^2=4$. The solution $(6, 8)$ of the latter equation gives the triad $(15, 9, -16)$ which satisfies (1). This triad yields the triads $(15, -16, 9)$, $(9, -16, 15)$ whence we get two sets for a, c, m, n and two forms:

$$\begin{aligned} -1, \quad -9, 1, 31&: \quad 2915M^2 - 3N^2 = 32;\\ -7, \ -15, 1, 25&: \ 13157M^2 - 21N^2 = 32. \end{aligned}$$

From these two binary forms infinitely many others can be generated, each of which will lead to solutions of (1).

4. Edgar [1] gives a solution of (1) corresponding in his notation to $k=16/3$; in my notation $a=16$, $c=13$. The corresponding equation (8) is

$$4N^2 - 391M^2 = 1$$

which has (as Edgar says) infinitely many solutions with even N, the smallest solution yielding

$$x = 8u + v, \qquad y = 8u - v, \qquad z = 13u$$

where $u=371133$, $v=1834670$. In the way described further forms can be generated from the permutations

$$(8u + v, -13u, 8u - v), \qquad (8u - v, -13u, 8u + v).$$

The pair $(a, c) = (10, 7)$ is also worthy of note: it leads to the form (6) which gives rise to infinitely many solutions of (1).

Two more examples can be given of small a, c:

$$a = 14, \qquad c = 11; \qquad a = 64, \qquad c = 61.$$

The first leads to the equation

$$7N^2 - 430M^2 = 2,$$

solvable infinitely often with N even, e.g. $N = 2124$, $M = 271$ whence

$$x = 2959, \qquad y = 835, \qquad z = -2981.$$

The second (64, 61) leads to the equation

$$(4N)^2 - 53815M^2 = 1$$

which is in fact solvably infinitely often with N *even* so that solutions of (1) are given by

$$x + y = 64M, \qquad x - y = N, \qquad z = -61M.$$

5. A difficult problem remains for consideration. Two solutions of (1) may be regarded as dependent if they can be connected by a finite number of binary forms as described above. Can simple criteria be determined for dependence? Can the independent solutions be completely specified?

I am grateful to Professor E. S. Barnes for some helpful comments.

Reference

1. H. M. Edgar, *Some remarks on the Diophantine equation* $x^3+y^3+z^3=x+y+z$, Proc. Amer. Math. Soc. **16** (1965), 148–153.

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