AN INEQUALITY FOR THE ELEMENTARY SYMMETRIC FUNCTIONS OF CHARACTERISTIC ROOTS

MARVIN MARCUS[1] AND HENRYK MINC[2]

In a recent paper [1] Frank obtained the following interesting result: if $A_1, \cdots, A_m$ are hermitian positive semidefinite n-square matrices and $\mu_1, \cdots, \mu_m$ are arbitrary complex numbers then

$$\left|\det\left(\sum_{i=1}^{m} \mu_i A_i\right)\right| \leqq \det\left(\sum_{i=1}^{m} |\mu_i| A_i\right). \tag{1}$$

Clearly any complex n-square matrix A can be expressed (not uniquely) in the form $A = \sum_{i=1}^{m} \mu_i A_i$. In the present paper we give an alternative, substantially simpler proof of Frank's theorem, and also discuss the case of equality.

For any n-square matrix X let $E_k(X)$ denote the kth elementary symmetric function of the characteristic roots of X.

THEOREM. *Let $A_1, \cdots, A_m$ be positive semidefinite hermitian n-square matrices and $\mu_1, \cdots, \mu_m$ be any complex numbers. Set*

$$A = \sum_{t=1}^{m} \mu_t A_t \tag{2}$$

and

$$S = \sum_{t=1}^{m} |\mu_t| A_t. \tag{3}$$

Then, for each $k = 1, \cdots, n$,

$$|E_k(A)| \leqq E_k(S). \tag{4}$$

In case $A_1, \cdots, A_m$ are positive definite, equality in (4) *can occur if and only if the numbers $\mu_1, \cdots, \mu_m$ are real nonnegative multiples of the same complex number.*

We first prove a preliminary result. Let $V_n(R)$ denote the space of complex n-tuples and for w_2 and w_1 in $V_n(R)$ let (w_1, w_2) denote the standard inner product in $V_n(R)$.

Presented to the Society, January 24, 1966; received by the editors July 16, 1965.

[1] This research was supported by the Air Force Office of Scientific Research, under Grant No. AFOSR-698-65.

[2] This research was supported by the Air Force Office of Scientific Research, under Grant No. AF-AFOSR-432-63.

LEMMA. *Let B be a complex n-square matrix and $v_1, \cdots, v_k$ be a fixed orthonormal set in $V_n(R)$. Let $C=(c_{ij})=((Bv_i, v_j))$ and suppose that*

$$|(Bu, u)| \leqq 1 \tag{5}$$

for all unit vectors u in $V_n(R)$. Then every characteristic root of C lies on or inside the unit circle.

PROOF. Let $x=(x_1, \cdots, x_m)$ be a unit vector in $V_m(R)$ and let

$$u_x = \sum_{j=1}^{m} x_j v_j \in V_n(R).$$

Since $\|x\|=1$ we have

$$\begin{aligned}(u_x, u_x) &= \left(\sum_{j=1}^{m} x_j v_j, \sum_{j=1}^{m} x_j v_j\right)\\ &= \sum_{i,j=1}^{m} x_i \bar{x}_j (v_i, v_j)\\ &= \sum_{j=1}^{m} |x_j|^2\\ &= 1.\end{aligned}$$

In other words, u_x is a unit vector in $V_n(R)$. Hence, by (5),

$$|(Bu_x, u_x)| \leqq 1. \tag{6}$$

We compute that

$$\begin{aligned}(Bu_x, u_x) &= \left(B\sum_{j=1}^{m} x_j v_j, \sum_{j=1}^{m} x_j v_j\right)\\ &= \sum_{i,j=1}^{m} x_i \bar{x}_j (Bv_i, v_j)\\ &= \sum_{i,j=1}^{m} x_i \bar{x}_j c_{ij}\\ &= (C\bar{x}, \bar{x}).\end{aligned}$$

Thus, from (6),

$$|(C\bar{x}, \bar{x})| \leqq 1. \tag{7}$$

Now choose $\bar{x}$ to be a unit characteristic vector of C corresponding to the characteristic root λ_t (of course, x is also a unit vector). Then (7) immediately implies

$$|\lambda_t| \leqq 1. \tag{8}$$

COROLLARY. *If C and B are the matrices defined in the Lemma then*

$$|\det(C)| \leqq 1. \tag{9}$$

We are now ready to prove our main result. Let $C_k(X)$ denote the kth (Grassmann) compound matrix of X [2, p. 16]. Then if $v_1, \cdots, v_k$ is any set of k vectors in $V_n(R)$ it is well known [2, p. 62] that

$$(C_k(X)v_1 \wedge \cdots \wedge v_k, v_1 \wedge \cdots \wedge v_k) = \det((Xv_i, v_j)) \tag{10}$$

where the indicated inner products are the standard inner products in the respective vector spaces.

PROOF OF THE THEOREM. For the purposes of proving (4) we may assume (by the standard continuity argument) that S is nonsingular and hence positive definite hermitian. Let $S^{-1/2}$ denote the positive definite square root of S^{-1} and set $B = S^{-1/2}AS^{-1/2}$. We prove that $|(Bu, u)| \leqq 1$ for any unit vector u. For, since the matrices $S^{-1/2}A_tS^{-1/2}$, $t = 1, \cdots, m$, are positive semidefinite, we have

$$\begin{aligned} |(Bu, u)| &= |(S^{-1/2}AS^{-1/2}u, u)| \\ &= \left|\left(S^{-1/2}\sum_{t=1}^{m} \mu_t A_t S^{-1/2}u, u\right)\right| \\ &= \left|\sum_{t=1}^{m} \mu_t (S^{-1/2}A_tS^{-1/2}u, u)\right| \\ &\leqq \sum_{t=1}^{m} |\mu_t| (S^{-1/2}A_tS^{-1/2}u, u) \\ &= \left(S^{-1/2}\sum_{t=1}^{m} |\mu_t| A_tS^{-1/2}u, u\right) \\ &= (S^{-1/2}SS^{-1/2}u, u) \\ &= (I_n u, u) \\ &= 1. \end{aligned} \tag{11}$$

It follows from the lemma that if $v_1, \cdots, v_k$ is any orthonormal set of vectors in $V_n(R)$ then

$$|\det((Bv_i, v_j))| \leqq 1. \tag{12}$$

Thus from (10) and (12) we see that

$$|(C_k(S^{-1/2}AS^{-1/2})v_1 \wedge \cdots \wedge v_k, v_1 \wedge \cdots \wedge v_k)| \leqq 1. \tag{13}$$

Let $e_1, \cdots, e_n$ be an orthonormal set of characteristic vectors of the positive definite hermitian matrix S corresponding respectively to the characteristic roots $\sigma_1, \cdots, \sigma_n$. Then, if $\omega=(\omega_1, \cdots, \omega_k)$, $1 \leqq \omega_1 < \cdots < \omega_k \leqq n$, is any increasing sequence of k integers chosen from $1, \cdots, n$, we have

$$
\begin{aligned}
&(C_k(S^{-1/2}AS^{-1/2})e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}, e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}) \\
&= (C_k(A)C_k(S^{-1/2})e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}, C_k(S^{-1/2})e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}) \\
&= (C_k(A)S^{-1/2}e_{\omega_1} \wedge \cdots \wedge S^{-1/2}e_{\omega_k}, S^{-1/2}e_{\omega_1} \wedge \cdots \wedge S^{-1/2}e_{\omega_k}) \\
&= (C_k(A)\sigma_{\omega_1}^{-1/2} \cdots \sigma_{\omega_k}^{-1/2} e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}, \sigma_{\omega_1}^{-1/2} \cdots \sigma_{\omega_k}^{-1/2} e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}) \\
&= \prod_{t=1}^{k} \sigma_{\omega_t}(C_k(A)e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}, e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}).
\end{aligned}
$$

Thus from (13)

$$
(C_k(A)e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}, e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}) \leqq \prod_{t=1}^{k} \sigma_{\omega_t}. \tag{14}
$$

Now, it is known that if $e_1, \cdots, e_n$ form an orthonormal basis of $V_n(R)$ then the $N=C_k^n$ Grassmann products $e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}$, $1 \leqq \omega_1 < \cdots < \omega_k \leqq n$, constitute an orthonormal basis of the space of N-tuples. Hence from (14) it follows that [2, p. 18]

$$
\begin{aligned}
|E_k(A)| &= |\operatorname{tr}(C_k(A))| \\
&= \left| \sum_{\omega} (C_k(A)e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}, e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}) \right| \\
&\leqq \sum_{\omega} |(C_k(A)e_{\omega_1} \wedge \cdots \wedge e_{\omega_k}, e_{\omega_1} \wedge \cdots \wedge e_{\omega_k})| \\
&\leqq \sum_{\omega} \prod_{t=1}^{k} \sigma_{\omega_t} \\
&= E_k(S).
\end{aligned} \tag{15}
$$

If equality holds in (15) then it must also hold in (11) and this in turn implies (in case $A_1, \cdots, A_m$ are positive definite and therefore $(S^{-1/2}A_tS^{-1/2}u, u) > 0$, $t=1, \cdots, m$,) that $\mu_1, \cdots, \mu_m$ have the same amplitude, i.e., they are nonnegative multiples of the same complex number μ. Conversely, suppose that $\mu_t = c_t\mu$, $c_t \geqq 0$, $t=1, \cdots, m$. Then

$$
|E_k(A)| = \left| E_k\left(\mu \sum_{t=1}^{m} c_tA_t\right)\right| = |\mu|^k E_k\left(\sum_{t=1}^{m} c_tA_t\right)
$$

$$= E_k\left(\sum_{t=1}^{m} |\mu| c_t A_t\right) = E_k(S).$$

COROLLARY. *Let A and S be defined as in the Theorem and let X_ω denote the k-square principal submatrix of X lying in rows*

$$\omega = (\omega_1, \cdots, \omega_k), \qquad 1 \leqq \omega_1 < \cdots \omega_k \leqq n.$$

Then

$$\left|\sum_{\omega} \det (A_\omega)\right| \leqq \sum_{\omega} \det (S_\omega). \tag{16}$$

In particular, for $k=1$ we have

$$|\operatorname{tr}(A)| \leqq \operatorname{tr}(S), \tag{17}$$

and for $k=n$

$$|\det(A)| \leqq \det(S). \tag{18}$$

The inequality (18) is due to Frank [1].

PROOF. These results follow immediately from the identity [2, p. 22]

$$E_k(X) = \sum_{\omega} \det (X_\omega).$$

REFERENCES

1. William M. Frank, *A bound on determinants*, Proc. Amer. Math. Soc. **16** (1965), 360–363.

2. Marvin Marcus and Henryk Minc, *A survey of matrix theory and matrix inequalities*, Allyn and Bacon, Boston, 1965.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA