ON THE INTEGRAL MODULI OF CONTINUITY IN L_p $(1<p<\infty)$ OF FOURIER SERIES WITH MONOTONE COEFFICIENTS

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1. **Introduction and results.** Let $f(x)$ be of period 2π and integrable L_p $(1<p<\infty)$. The integral moduli of continuity of first and second order of f in L_p are defined by

$$\omega_p(h;f) = \sup_{|t|\leqq h} \|f(x+t) - f(x)\|_p$$

and

$$\omega_p^*(h;f) = \sup_{0<t\leqq h} \|f(x+t) + f(x-t) - 2f(x)\|_p$$

respectively, where $\|\cdot\|_p$ denotes the norm in L_p. The Lipschitz and Zygmund classes Λ_p and Λ_p^* are then defined by $\omega_p(h;f)=O(h)$ and $\omega_p^*(h;f)=O(h)$ respectively.

The problem of what can be said about the integral modulus of continuity (of first order) of the functions of the class Λ_p^* $(1<p<\infty)$ was solved by A. Timan and M. Timan [6] for $p=2$ and in the general case by Zygmund [7] in the following way:[1]

$$f \in \Lambda_p^* \Rightarrow \omega_p(h;f) \leqq \begin{cases} A_p h\,|\log h|^{1/p} & \text{for } 1<p\leqq 2,\\ A_p h\,|\log h|^{1/2} & \text{for } 2\leqq p<\infty.\end{cases}$$

Both estimates are best possible in general. Here we shall show that the second estimate can be improved for a special class of functions.

THEOREM 1. *If $f\in L_p$ $(1<p<\infty)$ has a cosine or sine Fourier series with monotone coefficients, then*

$$f \in \Lambda_p^* \Rightarrow \omega_p(h;f) \leqq A_p h\,|\log h|^{1/p}.$$

The example of the function $f(x)=\sum_{n=1}^{\infty} n^{1/p-2}\cos nx$ $(1<p<\infty)$, which belongs to Λ_p^* and whose integral modulus $\omega_p(h;f)$ is $>C_p h|\log h|^{1/p}$. Zygmund [7] shows that the estimate of Theorem 1 is the best possible.

Recently Aljančić and Tomić [1] proved that if the sequence $\{\mu_n\}$ satisfies $\mu_n\geqq\mu_{n+1}\to 0$ and for a fixed p $(1<p<\infty)$

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[1] $A_p, B_p, \cdots$ denote constants which depend at most on p, but not necessarily always the same.

(1) $$\sum_{\nu=1}^{n-1} \nu^{1-1/p}\mu_\nu = O(n^{2-1/p}\mu_n) \quad \text{and} \quad \left\{\sum_{\nu=n}^{\infty} \nu^{p-2}\mu_\nu^p\right\}^{1/p} = O(n^{1-1/p}\mu_n),$$

then

(2) $$\omega_p(n^{-1}; f) \leqq A_p n^{1-1/p}\mu_n,$$

where f is the sum of either of the series

(3) $$\sum_{n=1}^{\infty} \mu_n \cos nx \quad \text{or} \quad \sum_{n=1}^{\infty} \mu_n \sin nx.$$

We shall prove here a more complete result:

THEOREM 2. *Let* $\{\mu_n\}$ *be a sequence which is monotonically decreasing to zero and such that for a fixed* p $(1<p<\infty)$

(4) $$\sum_{n=1}^{\infty} n^{p-2}\mu_n^p < \infty.$$

If f *is the sum of either of the series* (3), *then*

(5) $$\omega_p(n^{-1}; f) \leqq A_p n^{-1}\left\{\sum_{\nu=1}^{n-1} \nu^{2p-2}\mu_\nu^p\right\}^{1/p} + B_p\left\{\sum_{\nu=n}^{\infty} \nu^{p-2}\mu_\nu^p\right\}^{1/p}.$$

On account of A. Timan [5, p. 339]

$$\left\{\sum_{\nu=1}^{n-1} \nu^{2p-2}\mu_\nu^p\right\}^{1/p} \leqq A_p \sum_{\nu=1}^{n-1} \nu^{1-1/p}\mu_\nu,$$

the estimate (2) is included in that of (5). On the other hand, if $\mu_n = n^{-\alpha}$ with $\alpha > 1-1/p$,[2] both (2) and (5) give the same estimate

$$\omega_p(n^{-1}; f) = O(n^{1-1/p-\alpha}) = O(n^{1-1/p}\mu_n) \quad \text{when} \quad \alpha < 2 - 1/p,$$

but, for $\alpha = 2-1/p$, (2) cannot be applied because of (1_1), whereas (5) gives

$$\omega_p(n^{-1}; f) = O(n^{-1}\log^{1/p} n) = O(n^{1-1/p}\mu_n \log^{1/p} n).$$

As well as Theorem 1, the following theorem is partly based on a special case of Theorem 2.

THEOREM 3. *If* $\mu_n \geqq \mu_{n+1} \to 0$, *then*

(6) $$\sum_{n=1}^{\infty} n^{2p-2}\mu_n^p < \infty \quad \textit{for a fixed } p\ (1 < p < \infty)$$

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[2] $\alpha > 1-1/p$ is necessary to guarantee the convergence of the series in (4).

is a necessary and sufficient condition that the sum f of either of the series (3)

(i) *belongs to* Λ_p, *or*

(ii) *is equivalent to an absolutely continuous function whose derivative belongs to* L_p.

We remark that the results of Theorems 1–3 can be extended in an obvious manner to higher moduli and derivatives respectively. For example, for the modulus of order k, only the first term on the right side in (5) is to be replaced by

$$A_{p,k}n^{-k}\left\{\sum_{\nu=1}^{n-1}\nu^{(k+1)p-2}\mu_\nu^p\right\}^{1/p}.$$

2. **Proof of Theorem 2.** We note first that condition (4) is both necessary and sufficient that $f\in L_p$ [**8**, Chapter XII, Lemma 6.6]. We shall prove the theorem for the cosine series, the proof for the sine series being analogous.

On account of the symmetry of $f(x)$

$$\sup_{0<|t|\leqq h}\left\{\int_{-\pi}^{\pi}|f(x+t)-f(x)|^p\,dx\right\}^{1/p}$$
$$=\sup_{0<t\leqq h}\left\{\int_0^{\pi}|f(x-t)-f(x)|^p\,dx+\int_0^{\pi}|f(x+t)-f(x)|^p\,dx\right\}^{1/p},$$

the function of t in the braces on the right side being pair. Hence, it suffices to evaluate

$$I=\left\{\int_0^{\pi}|f(x\pm t)-f(x)|^p\,dx\right\}^{1/p}\quad\text{for } 0\ t\leqq h.$$

Let $h=\pi/2n$. Owing to $(a+b)^{1/p}\leqq a^{1/p}+b^{1/p}$ $(p>1)$, we have

$$\begin{aligned}I&\leqq\left\{\int_0^{\pi/n}|f(x\pm t)-f(x)|^p\,dx\right\}^{1/p}\\&\quad+\left\{\int_{\pi/n}^{\pi}|f(x\pm t)-f(x)|^p\,dx\right\}^{1/p}=I_1+I_2.\end{aligned}\tag{7}$$

By Minkowski's inequality

$$\begin{aligned}I_1&\leqq 2\left\{\int_0^{\pi/n}\left|\sum_{\nu=1}^{n-1}\mu_\nu\sin\tfrac{1}{2}\nu t\sin\nu(x\pm\tfrac{1}{2}t)\right|^p dx\right\}^{1/p}\\&\quad+\left\{\int_0^{\pi/n}\left|\sum_{\nu=n}^{\infty}\mu_\nu[\cos\nu(x\pm t)-\cos\nu x]\right|^p dx\right\}^{1/p}=I_{11}+I_{12}.\end{aligned}\tag{8}$$

As, by Hölder's inequality,

$$\sum_{\nu=1}^{n-1} \nu\mu_\nu \leqq A_p n^{1/p} \left\{ \sum_{\nu=1}^{n-1} \nu^{2p-2} \mu_\nu^p \right\}^{1/p},$$

we get

$$I_{11} \leqq t \left\{ \int_0^{\pi/n} \left(\sum_{\nu=1}^{n-1} \nu\mu_\nu \right)^p dx \right\}^{1/p} \leqq A_p n^{-1} \left\{ \sum_{\nu=1}^{n-1} \nu^{2p-2} \mu_\nu^p \right\}^{1/p}. \tag{9}$$

For the latter of the integrals in (8) we find in virtue of $t \leqq \pi/2n$

$$\begin{aligned} I_{12} &\leqq \left\{ \int_{\pm t}^{\pi/n \pm t} \left| \sum_{\nu=n}^{\infty} \mu_\nu \cos \nu x \right|^p dx \right\}^{1/p} \\ &\quad + \left\{ \int_0^{\pi/n} \left| \sum_{\nu=n}^{\infty} \mu_\nu \cos \nu x \right|^p dx \right\}^{1/p} \\ &\leqq (2^{1/p} + 1) \left\{ \int_0^{3\pi/2n} \left| \sum_{\nu=n}^{\infty} \mu_\nu \cos \nu x \right|^p dx \right\}^{1/p} \\ &\leqq 3 \left\{ \sum_{m=n}^{\infty} \int_{3\pi/2(m+1)}^{3\pi/2m} \left| \sum_{\nu=n}^{\infty} \mu_\nu \cos \nu x \right|^p dx \right\}^{1/p}. \end{aligned}$$

As, for $3\pi/2(m+1) \leqq x \leqq 3\pi/2m$ $(m = n, n+1, \cdots)$,

$$\left| \sum_{\nu=n}^{\infty} \mu_\nu \cos \nu x \right| \leqq \sum_{\nu=n}^{m} \mu_\nu + \pi x^{-1} \mu_{m+1} \leqq \sum_{\nu=n}^{m} \mu_\nu + \tfrac{2}{3}(m+1)\mu_{m+1},$$

we see that

$$I_{12}^p \leqq A_p \sum_{m=n}^{\infty} m^{-2} \left(\sum_{\nu=n}^{m} \mu_\nu \right)^p + B_p \sum_{m=n}^{\infty} m^{p-2} \mu_m^p.$$

Hardy's inequality [**3**, Chapter IX, Miscellaneous theorems and examples 346]

$$\sum_{m=1}^{\infty} m^{-2} \left(\sum_{\nu=1}^{m} C_\nu \right)^p \leqq K_p \sum_{m=1}^{\infty} m^{p-2} C_m^p \qquad (C_m \geqq 0,\ p > 1), \tag{10}$$

with $C_m = 0$ for $m < n$ and $C_m = \mu_m$ for $m \geqq n$, shows that the first of these sums is majorized by the latter. Hence,

$$I_{12} \leqq B_p \left\{ \sum_{\nu=n}^{\infty} \nu^{p-2} \mu_\nu^p \right\}^{1/p}. \tag{11}$$

If $D_\nu(x)$ denotes the Dirichlet kernel, an Abel transformation combined with Minkowski's inequality gives

$$
\begin{aligned}
I_2 \leqq & \left\{ \int_{\pi/n}^{\pi} \left| \sum_{\nu=1}^{n} \Delta\mu_\nu [D_\nu(x \pm t) - D_\nu(x)] \right|^p dx \right\}^{1/p} \\
& + \left\{ \int_{\pi/n}^{\pi} \left| \sum_{\nu=n+1}^{\infty} \Delta\mu_\nu [D_\nu(x \pm t) - D_\nu(x)] \right|^p dx \right\}^{1/p} = I_{21} + I_{22}.
\end{aligned}
\tag{12}
$$

By dividing the interval $(\pi/n, \pi)$ in subintervals $(\pi/(m+1), \pi/m)$ $(m=1, \cdots, n-1)$ and applying $D_\nu'(x)=O(\nu^2)$ for $0 \leqq x \leqq \pi$ and $D_\nu'(x)=O(x^{-2})+O(\nu x^{-1})=O(\nu x^{-1})$ for $\pi/\nu \leqq x \leqq \pi$, one obtains in such a subinterval

$$
\begin{aligned}
&\left| \sum_{\nu=1}^{n} \Delta\mu_\nu [D_\nu(x \pm t) - D_\nu(x)] \right| \\
&\quad \leqq t \left(\sum_{\nu=1}^{m} + \sum_{\nu=m+1}^{n} \right) \Delta\mu_\nu \left| D_\nu'(x + \theta_\nu t) \right| \\
&\quad = O(t) \sum_{\nu=1}^{m} \nu^2 \Delta\mu_\nu + O(t)(x-t)^{-1} \sum_{\nu=m+1}^{n} \nu \Delta\mu_\nu, \qquad (-1 < \theta_\nu < 1)
\end{aligned}
$$

because, on account of $x \geqq \pi/(m+1)$, the second estimate for $D_\nu'(x)$ is applicable to every member in the latter sum. If we remember that by Abel transformation

$$
\sum_{\nu=1}^{m} \nu^2 \Delta\mu_\nu \leqq 2 \sum_{\nu=1}^{m} \nu\mu_\nu, \qquad \sum_{\nu=m+1}^{n} \nu \Delta\mu_\nu \leqq \sum_{\nu=m+1}^{n} \mu_\nu + m\mu_{m+1},
$$

and observe that, owing to $t \leqq \pi/2n$, the inequality $(x-t)^{-1} \leqq 2x^{-1}$ $(x \geqq 2t)$ may be applied in any of the mentioned subintervals, we get at last the following estimate:

$$
\begin{aligned}
&\left| \sum_{\nu=1}^{n} \Delta\mu_\nu [D_\nu(x \pm t) - D_\nu(x)] \right| \\
&\qquad = O(t) \sum_{\nu=1}^{m} \nu\mu_\nu + O(tm) \sum_{\nu=m+1}^{n} \mu_\nu + O(tm^2\mu_m).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_{21}^p &= \sum_{m=1}^{n-1} \int_{\pi/(m+1)}^{\pi/m} \left| \sum_{\nu=1}^{n} \Delta\mu_\nu [D_\nu(x \pm t) - D_\nu(x)] \right|^p dx \\
&= O(t^p) \left\{ \sum_{m=1}^{n-1} m^{-2} \left(\sum_{\nu=1}^{m} \nu\mu_\nu \right)^p + \sum_{m=1}^{n-1} m^{p-2} \left(\sum_{\nu=m+1}^{n} \mu_\nu \right)^p + \sum_{m=1}^{n-1} m^{2p-2} \mu_m^p \right\}.
\end{aligned}
$$

By Hardy's inequality (10) with $C_m = m\mu_m$ for $m<n$ and $C_m=0$ for $m \geqq n$, the first of these sums is essentially majorized by the third. The same holds for the second sum according to another inequality of Hardy [**3**, ibid.]:

$$\sum_{m=1}^{\infty} m^{-c}\left(\sum_{\nu=m}^{\infty} C_\nu\right)^p \leqq K_p \sum_{m=1}^{\infty} m^{p-c} C_m^p \qquad (c<1, C_m \geqq 0, p>1),$$

if we choose $c=2-p$ and $C_m=\mu_{m+1}$ for $m<n$ and $C_m=0$ for $m \geqq n$. Hence,

$$I_{21} \leqq A_p n^{-1}\left\{\sum_{\nu=1}^{n-1} \nu^{2p-2}\mu_\nu^p\right\}^{1/p}. \tag{13}$$

Lastly,

$$\begin{aligned} I_{22} &\leqq 2\left\{\int_{\pi/2n}^{\pi+\pi/2n}\left|\sum_{\nu=n+1}^{\infty}\Delta\mu_\nu D_\nu(x)\right|^p dx\right\}^{1/p} \\ &= O(\mu_n)\left\{\int_{\pi/2n}^{\infty} x^{-p}\,dx\right\}^{1/p} = O(n^{1-1/p}\mu_n). \end{aligned}$$

As

$$\left\{\sum_{\nu=1}^{n-1}\nu^{2p-2}\mu_\nu^p\right\}^{1/p} \geqq \mu_{n-1}\left\{\sum_{\nu=1}^{n-1}\nu^{2p-2}\right\}^{1/p} \geqq C_p n^{2-1/p}\mu_n,$$

one finds

$$I_{22} \leqq A_p n^{-1}\left\{\sum_{\nu=1}^{n-1}\nu^{2p-2}\mu_\nu^p\right\}^{1/p}. \tag{14}$$

Collecting in (8) and (12) the estimates (9), (11), (13) and (14), from (7) follows Theorem 2.

3. **Proof of Theorem 1.** Recently, Konjuškov [**4**] called attention to the fact that if $f \in L_p$ $(1<p<\infty)$ has a cosine or sine series with monotone coefficients,[3] then

$$\omega_p^*(n^{-1}; f) \geqq C_p n^{1-1/p}\mu_n \qquad (C_p>0). \tag{15}$$

Konjuškov deduced (15) from his results about the relationship between the best trigonometric approximation of f in L_p and the Fourier coefficients of f. As (15), together with a special case of Theorem 2, is essentially in the proof of Theorem 1, we give here a direct

[3] He even supposed that for a fixed $\tau>0$, the sequence $n^{-\tau}\mu_n$ is only almost decreasing. His result is not limited to the modulus of second order only.

proof of (15). It is based on the following identity, easily verified:

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}[2f(x)-f(x+t)-f(x-t)]T_{m,n}(x)\,dx=\sum_{\nu=m}^{n}\mu_\nu\sin^2\tfrac{1}{2}\nu t, \tag{16}$$

where $T_{m,n}(x)=\sum_{\nu=m}^{n}\cos\nu x$ and f is a cosine series. If we set $t=\pi/n$ in (16) and choose $m=[n/2]$, then

$$\sum_{\nu=m}^{n}\mu_\nu\sin^2\frac{\nu\pi}{2n}\geqq\frac{1}{n^2}\sum_{\nu=m}^{n}\nu^2\mu_\nu\geqq\frac{m^2}{n^2}\mu_n(n-m+1)\geqq Cn\mu_n.$$

On the other hand, in virtue of Hölder's inequality,

$$\begin{aligned}\frac{1}{4\pi}\int_{-\pi}^{\pi}&[2f(x)-f(x+\pi/n)-f(x-\pi/n)]T_{m,n}(x)\,dx\\&\leqq A_pn^{1/p}\left\{\int_{-\pi}^{\pi}|f(x+\pi/n)+f(x-\pi/n)-2f(x)|^pdx\right\}^{1/p}\\&\leqq A_pn^{1/p}\omega_p^*(\pi/n;f),\end{aligned}$$

because $(1/q=1-1/p)$

$$\int_{-\pi}^{\pi}|T_{m,n}(x)|^q\,dx=2\left\{\int_0^{\pi/n}O(n^q)\,dx+\int_{\pi/n}^{\pi}O(x^{-q})\,dx\right\}=O(n^{q-1}).$$

Hence, (15) follows.

The proof of Theorem 1 is now immediate. If μ_n are the coefficients of f and $\omega_p^*(n^{-1};f)\leqq A_pn^{-1}$, then, according to (15), $\mu_n\leqq B_pn^{-2+1/p}$ and one has only to apply Theorem 2 in this special case.

4. **Proof of Theorem 3.** On account of the well-known theorem of Hardy and Littlewood [2], which asserts the equivalence of (i) and (ii) in the general case, we have only to prove that (6)⇒(i) and that (ii)⇒(6).

(6)⇒(i). Because of $s_n=\sum_{\nu=1}^{n}\nu^{2p-2}\mu_\nu^p<\infty$, an Abel transformation gives

$$\sum_{\nu=n}^{\infty}\nu^{p-2}\mu_\nu^p=\sum_{\nu=n}^{\infty}\nu^{-p}(s_\nu-s_{\nu-1})=O(n^{-p}).$$

Hence, the second term in (5) is also of order $O(n^{-1})$, i.e. $f\in\Lambda_p$.

(ii)⇒(6). The proof proceeds, with necessary changes, along the same lines as the necessity part in the proof of Lemma 6.6 in [8, Chapter XII], and is adapted for the sine series.

Let

$$F(x) = \int_0^x f(t)\,dt = \sum_{n=1}^{\infty} n^{-1}\mu_n\, 1 - \cos nx.$$

Then, even simpler than in the proof of the cited lemma, $F(\pi/n) \geqq C\mu_n$. If we set

$$G(x) = \int_0^x dt \int_0^t |f'(u)|\,du.$$

then $F(x) \leqq G(x)$. Hence, applying twice Hardy's inequality [8, Chapter I, p. 20], we get

$$\sum_{n=2}^{\infty} n^{2p-2}\mu_n^p \leqq A_p \sum_{n=2}^{\infty} n^{2p-2} G^p(\pi/n) \leqq A_p \sum_{n=2}^{\infty} \int_{\pi/n}^{\pi/(n-1)} \left[\frac{G(x)}{x}\right]^p x^{-p}\,dx$$

$$= A_p \int_0^{\pi} \left[\frac{G(x)}{x}\right]^p x^{-p}\,dx \leqq A_p \int_0^{\pi} \left(\int_0^x |f'(t)|\,dt\right)^p x^{-p}\,dx$$

$$\leqq A_p \int_0^{\pi} |f'(x)|^p\,dx < \infty,$$

which completes the proof of Theorem 3.

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