ON THE ZEROS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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This note gives a proof of the following theorem:

In the differential equation

$$y'' + p(x)y = 0, \tag{1}$$

where primes mean derivatives, suppose $p(x)$ to be positive or zero, monotonic and concave (no point of an arc lies below its chord) in some closed interval $[a, b]$. If

$$\int_a^b p(x)\,dx \geqq (9/8)n^2\pi^2/(b-a), \tag{2}$$

where n is an integer, then every solution of (1) has at least n zeros in $[a, b]$. The number $9/8$ cannot be replaced by a smaller one.

A theorem similar to this, but with more restrictive hypotheses, has been proved by Makai [**5**].

Related theorems have been proved by a number of authors, back as far as Liouville. More recent examples are given in the references.

The proof depends on three lemmas.

LEMMA 1. *If the equation*

$$y'' + q(x)y = 0, \tag{3}$$

where $q(x)$ is continuous, has a solution with consecutive zeros at $x=c$ and $x=d$, and if

$$\int_c^d q(x)\cos(2\pi(x-c)/(d-c))\,dx \leqq 0, \tag{4}$$

then

$$\int_c^d q(x)\,dx \leqq \pi^2/(d-c). \tag{5}$$

PROOF. Let $y(x)$ be the solution referred to, and let

$$z(x) = (2/(d-c))^{1/2}\sin(\pi(x-c)/(d-c)), \quad \text{so that} \quad \int_c^d z^2\,dx = 1.$$

Now

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$$0 \leqq \int_c^d ((z'y - zy')^2/y^2)\, dx = \int_c^d (z'y - zy')(z/y)'\, dx$$

$$= (z'y - zy')(z/y)\Big|_c^d - \int_c^d (z/y)(z''y - zy'')\, dx$$

$$= \int_c^d (\pi^2/(d - c)^2 - q(x))z^2\, dx,$$

from (3). Since $y(x)$ and $z(x)$ have simple zeros at $x = c$ and $x = d$, their ratio has a limit at each end of the interval, so the integrated term vanishes. Then

$$\pi^2/(d - c)^2 \geqq \int_c^d q(x)z^2\, dx$$

$$= (1/(d - c)) \int_c^d q(x)[1 - \cos(2\pi(x - c)/(d - c))]\, dx.$$

From this and (4), (5) follows.

COROLLARY. *The number s is not less than d, if s is determined by*

$$\int_c^s q(x)\, dx = \pi^2/(s - c).$$

LEMMA 2. *If $p(x)$ is concave, then*

$$\int_c^d p(x) \cos(2\pi(x - c)/(d - c))\, dx \leqq 0.$$

This lemma is due to E. Makai [**4**], pp. 370–371.

LEMMA 3. *Let $\{x_i\}$ and $\{x_i'\}$, $i = 0, 1, 2, \cdots, n$, be two sets of numbers such that*

(a) $x_0 < x_1 < x_2 < \cdots < x_n$, $x_0' < x_1' < x_2' < \cdots < x_n'$;

(b) $x_1 - x_0 \geqq x_2 - x_1 \geqq x_3 - x_2 \geqq \cdots \geqq x_n - x_{n-1}$,

and similarly for the $\{x_i'\}$;

(c) $x_i' \leqq x_i$, $i = 1, 2, \cdots, n-1$;

(d) $x_0 = x_0'$, $x_n = x_n'$.

Then

$$\sum_{i=1}^{n} 1/(x_i' - x_{i-1}') \leqq \sum_{i=1}^{n} 1/(x_i - x_{i-1}). \tag{6}$$

PROOF. The case $n = 1$ is trivial, and if $n = 2$ the proof is elementary.

To complete the proof by induction, let S_n' and S_n stand for the left and right members of (6) respectively. Then $S_n' \leqq S_n$ must be shown to imply $S_{n+1}' \leqq S_{n+1}$, with n replaced by $n+1$ in (a), (b), (c) and (d). Let g be the least of $x_i - x_i'$, and let $x_i'' = x_i' + g$, $i = 1, 2, \cdots, n$, so that $x_i'' - x_{i-1}'' = x_i' - x_{i-1}'$. Let S_{n+1}'' be formed from S_{n+1}' by substituting x_i'' for x_i'. ($x_0'' = x_0' = x_0$, $x_{n+1}'' = x_{n+1}' = x_{n+1}$.) Then

$$\begin{aligned} S_{n+1}'' - S_{n+1}' &\\ &= 1/(x_{n+1} - x_n'') - 1/(x_{n+1} - x_n') + 1/(x_1'' - x_0) - 1/(x_1' - x_0) \\ &= g[1/(x_{n+1} - x_n'')(x_{n+1} - x_n') - 1/(x_1'' - x_0)(x_1' - x_0)]. \end{aligned}$$

Now $x_{n+1} - x_n'' = x_{n+1} - x_n + (x_n - x_n') - g \geqq x_{n+1} - x_n > 0$, and $x_{n+1} - x_n'' \leqq x_{n+1} - x_n' \leqq x_1' - x_0$ by (c) above; while $x_1'' - x_0 \geqq x_1' - x_0$. Hence the first denominator in the square bracket is not greater than the second, so that the bracket is positive or zero. Then $S_{n+1}'' - S_{n+1}' \geqq 0$.

But for at least one value of i, say $i = k$, $0 < k < n+1$, $x_k'' = x_k$. Then S_{n+1} and S_{n+1}'' can each be broken into two sums, from $i = 1$ to $i = k$ and from $i = k+1$ to $i = n+1$ respectively. Each of these latter sums contains no more than n terms and satisfies the hypotheses of the lemma. Hence the induction hypothesis applies to each, and their addition yields $S_{n+1} - S_{n+1}'' \geqq 0$. Addition of the previous inequality gives the lemma. (This proof, by E. Makai, Jr., was kindly sent to the author by the referee.)

Proof of the theorem. Consider first that $p(x)$ has the special form

$$r(x) = 2An^2\pi^2 x,$$

where A is a positive constant and $[a, b]$ is $[0, 1]$. Inequality (4) is satisfied, and it will appear later that A can be 9/8. Choose a solution $y(x)$ with $y(0) = 0$. Successive applications of Lemma 1 show that the succeeding zeros of $y(x)$ precede respectively the numbers $x_1, x_2, \cdots$, where the x_i are determined by the equations

$$\begin{aligned} &\int_0^{x_1} r(x)\,dx = \pi^2/x_1 \quad \text{or} \quad An^2 x_1^3 = 1, \\ (7) \qquad &\int_{x_1}^{x_2} r(x)\,dx = \pi^2/(x_2 - x_1) \quad \text{or} \quad An^2(x_2^2 - x_1^2)(x_2 - x_1) = 1, \text{ etc.} \end{aligned}$$

For $n = 1$, the theorem is true by Lemma 1. Assume it true for n. For $n+1$, the points X_i corresponding to the x_i are $X_i = (n/(n+1))^{2/3} x_i$, as is seen by substitution. The theorem will be true for $n+1$ if

$$A(n+1)^2(X_{n+1}^2 - X_n^2)(X_{n+1} - X_n) = 1$$

while $X_{n+1} \leqq 1$, or if

$$A(n+1)^2\{1-(n/(n+1))^{4/3}x_n^2\}\{1-(n/(n+1))^{2/3}x_n\} \geqq 1.$$

Since $x_n \leqq 1$, A can be chosen so that

$$A(n+1)^2\{1-(n/(n+1))^{4/3}\}\{1-(n/(n+1))^{2/3}\} \geqq 1,$$

or

$$A\{(n+1)^{2/3}-n^{2/3}\}^2\{(n+1)^{2/3}+n^{2/3}\} \geqq 1.$$

Let the part in braces be $f(n)$. Then $f(1) > 0.8914$, since $2^{2/3} > 1.587$. Then if $A = 9/8$, $Af(1) > 1$.

The behavior of $f(n)$ for large values of n can be examined by treating n as a continuous variable, increasing without limit. It can be shown by elementary arguments that f'/f is negative and that the limit of $f(n)$ as n increases is $8/9$. Hence if $A = 9/8$ the theorem is true for the function $r(x)$ considered.

Now let $p(x)$ be a function different from $r(x)$, and satisfying the hypotheses of the theorem. Since $\int_0^1 p(x)dx \geqq (9/8)n^2\pi^2$, $p(x)$ must be greater than $r(x)$ near $x=0$ and less (perhaps) near $x=1$. Suppose first that $p(x)$ is increasing. The equations (7), with $p(x)$ for $r(x)$, will determine numbers x_1', x_2', $\cdots$, which are not less than the respective zeros, by Lemma 1, and such that $x_1' \geqq x_2' - x_1' \geqq \cdots$.

Now $x_i' \leqq x_i$, $i = 1, 2, \cdots, n$. For if not, let $x_i' \leqq x_i$, $i = 1, 2, \cdots, j-1$, and suppose $x_j' > x_j$. Then

$$\begin{aligned} (1/\pi^2)\int_0^{x_j'} p(x)\,dx &= 1/x_1' + 1/(x_2' - x_1') + \cdots + 1/(x_j' - x_{j-1}') \\ &< 1/x_1' + 1/(x_2' - x_1') + \cdots + 1/(x_j - x_{j-1}') \\ &\leqq 1/x_1 + 1/(x_2 - x_1) + \cdots + 1/(x_j - x_{j-1}) \end{aligned} \tag{8}$$

by Lemma 3, and this last sum is $(1/\pi^2)\int_0^{x_j} r(x)dx$. But if $F(x) = \int_0^x (p(t) - r(t))dt$, $F'(x) = p(x) - r(x)$, which is zero at no more than one point between 0 and 1, by the concavity of $p(x)$. Hence $F(x)$ has at most one maximum, and is positive between 0 and 1. Since $p(x)$ is positive, $\int_0^{x_j''} p(x)dx < \int_0^{x_j'} p(x)dx < \int_0^{x_j''} r(x)dx$ by (8), so $F(x_j) < 0$, a contradiction. Hence $x_j' \leqq x_j$, $j = 1, 2, \cdots, n$. The inequality above will apply directly if $n=1$. Sturm's separation theorem shows that every other solution will have at least n zeros. A linear change of variable from $[0, 1]$ to $[a, b]$ does not affect the argument. If $p(x)$ is decreasing, the same proof can be used from right to left.

That $9/8$ is the best possible constant can be shown thus: The solution of $y'' + xy = 0$ that vanishes at the origin is $y(x) = (x)^{1/2}J_{1/3}(2x^{3/2}/3)$, where $J(\ )$ is the Bessel's function of the first kind. If $t = 2x^{3/2}/3$, the function $Y(t) = (t)^{1/2}J_{1/3}(t)$ has zeros at points corresponding to those of $y(x)$, and satisfies the equation

$$d^2Y/dt^2 + (1 + 5/(36t^2))Y = 0. \tag{9}$$

The zeros t_n of $Y(t)$ after the first will have the form

$$t_n = n\pi + h + o(1/n),$$

where h is some constant, since the interval between successive zeros approaches π as n becomes infinite. If a constant $B < 9/8$ could be used in (2), that inequality would show that the number v_n, determined by $\int_0^{v_n} x\,dx = Bn^2\pi^2/v_n$, was not less than x_n, the nth positive zero of $J_{1/3}(2x^{3/2}/3)$. This would imply

$$\begin{aligned} B &= v_n^3/2n^2\pi^2 \geqq x_n^3/2n^2\pi^2 = (3t_n/2)^2/2n^2\pi^2 \\ &= (9/8n^2\pi^2)(n\pi + h + o(1/n))^2, \end{aligned}$$

which approaches 9/8. This completes the proof.

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