CHARACTER KERNELS OF DISCRETE GROUPS[1]

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Let G be an arbitrary discrete group and let $\Gamma = \mathbf{C}[G]$ be its group algebra over the complex numbers $\mathbf{C}$. If $\mathfrak{R}$ is an irreducible representation of the algebra then $\mathfrak{R}(\Gamma) = P$ is primitive and hence isomorphic to a dense set of linear transformations over D, the commuting ring of $\mathfrak{R}$ [**4**, p. 28]. Let L be the center of D. If $\dim_L P < \infty$ then we say that $\mathfrak{R}$ is finite and since P is central simple over L [**4**, p. 122] we have $\dim_L P = m^2$. We set $m = \deg \mathfrak{R}$, the degree of $\mathfrak{R}$. If G is finite then $\mathbf{C}$ is always the commuting ring of $\mathfrak{R}$ so this agrees with the usual definition of degree.

Again let $P = \mathfrak{R}(\Gamma)$. Then by a theorem of Amitsur [**1**] $\deg \mathfrak{R} \leqq n$ if and only if for every $2n$ elements $x_1, \cdots, x_{2n}$ in P we have

$$[x_1, \cdots, x_{2n}] = \sum \pm x_{i_1} x_{i_2} \cdots x_{i_{2n}} = 0.$$

The above is known as the standard identity of degree $2n$. For infinite discrete groups, representation theory is not particularly well behaved. Therefore we will make use of these identities in $\mathbf{C}[G]$.

If $g \in G$ we say that g is in the kernel of $\mathfrak{R}$ if and only if $\mathfrak{R}(g) = \mathfrak{R}(1) = 1$. We set $\mathfrak{K}_n(G) = \bigcap \ker \mathfrak{R}$ where $\mathfrak{R}$ runs over all irreducible representations of degree $> n$. We study groups G with $\mathfrak{K}_n(G) > 1$. It is convenient to let $b(G) =$ lub of the degrees of the irreducible representations of G. If $b(G) \leqq n$ then trivially $\mathfrak{K}_n(G) = G$. Thus we will be interested mainly in groups with $b(G) > n$.

THEOREM 1. *Let $I_n = I_n[G]$ be the linear subspace of $\mathbf{C}[G]$ spanned by all terms of the form $[x_1, \cdots, x_{2n}]$ with $x_i \in \mathbf{C}[G]$. Then $g \in \mathfrak{K}_n(G)$ if and only if $(1-g)I_n = 0$.*

PROOF. First let $g \in \mathfrak{K}_n(G)$. Let $\mathfrak{R}$ be any irreducible representation of $\mathbf{C}[G]$ and consider $\mathfrak{R}((1-g)I_n)$. If $\deg \mathfrak{R} > n$ then $\mathfrak{R}(1-g) = 0$. If $\deg \mathfrak{R} \leqq n$ then $\mathfrak{R}(I_n) = 0$. Hence in either case $\mathfrak{R}((1-g)I_n) = 0$. Since this holds for all $\mathfrak{R}$ and $\mathbf{C}[G]$ is semi simple [**5**, Theorem 5.2] this yields $(1-g)I_n = 0$.

Conversely let $(1-g)I_n = 0$. Let $J_n = \{a \in \mathbf{C}[G] \mid aI_n = 0\}$ so that J_n is clearly a left ideal of $\mathbf{C}[G]$. To show that it is a right ideal we need only show for $h \in G$ that $J_n h \subseteq J_n$. Since clearly $h^{-1} I_n h = I_n$ we have

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$$(J_nh)I_n = (J_nh)(h^{-1}I_nh) = J_nI_nh = 0$$

and so J_n is a two-sided ideal. Let $\mathfrak{R}$ be an irreducible representation of $\mathbf{C}[G]$ of degree $>n$. Let V be the corresponding left $\mathbf{C}[G]$-module and set $V^* = \{v \in V \mid J_nv = 0\}$. Since J_n is a right ideal, V^* is a submodule of V. Hence either $V^* = V$ or $V^* = 0$. Now deg $\mathfrak{R} > n$ so $\mathfrak{R}(\mathbf{C}[G])$ does not satisfy the standard identity of degree $2n$. Hence $\mathfrak{R}(I_n) \neq 0$ and $I_nV \neq 0$. But $J_n(I_nV) = 0$ so $I_nV \subseteq V^*$ and hence $V^* = V$. Since $(1-g) \in J_n$ and $J_nV = 0$ we have $\mathfrak{R}(g) = \mathfrak{R}(1)$ and the result follows.

Now $I_n[G]$ is spanned as a linear space over $\mathbf{C}$ by all terms of the form $[g_1, \cdots, g_{2n}]$ with $g_i \in G$. Hence we have the following.

Corollary 2. *Let $b(G) > n$. Then $|\mathfrak{K}_n(G)| \leqq \frac{1}{2}(2n)!$.*

Proof. Now $b(G) > n$ implies that $I_n[G] \neq 0$. Thus we can find $g_i \in G$ with $\beta = [g_1, \cdots, g_{2n}] \neq 0$. If $g \in \mathfrak{K}_n(G)$ then $(1-g)\beta = 0$ so that $g\beta = \beta$. If $\beta = \sum c_ih_i$ with $c_i \in \mathbf{C}$ and $h_i \in G$ then $\mathfrak{K}_n(G)$ permutes all those h_i with $c_i > 0$. Thus $|\mathfrak{K}_n(G)|$ divides r, the number of such h_i. Since $\beta \neq 0$ and $\sum c_i = 0$ we see that $1 \leqq r \leqq \frac{1}{2}(2n)!$ and the result follows.

If H is a subgroup of G then $\mathbf{C}[H]$ is naturally embedded in $\mathbf{C}[G]$. Moreover in this embedding $I_n[H] \subseteq I_n[G]$. With this remark we have

Corollary 3. *Let H be a subgroup of G with $b(H) > n$. Then $\mathfrak{K}_n(G) \subseteq \mathfrak{K}_n(H)$.*

Proof. Since $b(H) > n$ there exists $h_i \in H$ with $\beta = [h_1, \cdots, h_{2n}] \neq 0$. Let $g \in \mathfrak{K}_n(G)$. Since $\beta = g\beta$ we see clearly that $g \in \langle h_1, \cdots, h_{2n}\rangle \subseteq H$. Now $(1-g)I_n[G] = 0$ implies $(1-g)I_n[H] = 0$ so $g \in \mathfrak{K}_n(H)$.

The following result essentially reduces the study of groups G with $b(G) > n$ and $\mathfrak{K}_n(G) > 1$ to a study of finite groups.

Theorem 4. *Let $b(G) > n$ and $\mathfrak{K}_n(G) > 1$. Let $\beta_1, \cdots, \beta_m$ be a finite number of nonzero elements of $\mathbf{C}[G]$. Then we can find subgroups H and N of G such that*

(i) $\beta_1, \cdots, \beta_m \in \mathbf{C}[H]$;

(ii) $b(H) > n$;

(iii) $\mathfrak{K}_n(H) = \mathfrak{K}_n(G)$;

(iv) *N is a normal subgroup of H with $\overline{H} = H/N$ finite and $b(\overline{H}) > n$;*

(v) *under the natural homomorphism $\mathbf{C}[H] \to \mathbf{C}[\overline{H}]$ we have $\beta_i \to \bar{\beta}_i \neq 0$ and $\mathfrak{K}_n(G) \simeq [\mathfrak{K}_n(G)]^- = \mathfrak{K}_n(\overline{H})$.*

Proof. First we show that we can find group elements g_{ij} with

$i=1, 2, \cdots, r$ and $j=1, 2, \cdots, 2n$ such that $g\in\mathfrak{K}_n(G)$ if and only if for all i

$$(1-g)[g_{i,1}, g_{i,2}, \cdots, g_{i,2n}] = 0.$$

Let $\mathcal{S}$ be the set of all terms $\alpha=[g_1, \cdots, g_{2n}]\neq 0$ with $g_i\in G$. For each such $\alpha\in\mathcal{S}$ set $\mathfrak{K}^\alpha=\{g\in G\mid (1-g)\alpha=0\}$. As in the proof of Corollary 2, $|\mathfrak{K}^\alpha|\leqq\frac{1}{2}(2n)!$. By Theorem 1, $\mathfrak{K}_n(G)=\bigcap\mathfrak{K}^\alpha$. Since each $\mathfrak{K}^\alpha$ is finite, clearly only a finite intersection is required.

For the remainder of the proof fix such a set $\{g_{ij}\}\subseteq G$. Let H be a finitely generated subgroup of G with $H\supseteq\{g_{ij}\}$ and $\beta_1, \cdots, \beta_m \in \mathbf{C}[H]$. Such groups clearly exist. Clearly $b(H)>n$ and we have $\mathfrak{K}_n(H)\supseteq\mathfrak{K}_n(G)$. But if $h\in\mathfrak{K}_n(H)$ then $(1-h)[g_{i,1}, \cdots, g_{i,2n}]=0$ for all i so $h\in\mathfrak{K}_n(G)$. Hence $\mathfrak{K}_n(H)=\mathfrak{K}_n(G)$. With this choice of H we have (i), (ii) and (iii) of Theorem 4 satisfied.

We now show that H is a subdirect product of finite groups. Fix $g\in\mathfrak{K}_n(H)$ with $g\neq 1$. Let h be any nonidentity element of H. In $\mathbf{C}[H]$ the expression $\gamma=(1-g)(1-h)$ is nonzero since otherwise $1+gh=g+h$ and so $1=g$ or h. Since $\mathbf{C}[H]$ is semisimple there exists an irreducible representation $\mathfrak{R}$ of $\mathbf{C}[H]$ with $\mathfrak{R}(\gamma)\neq 0$. Hence $\mathfrak{R}(g)\neq 1$ and $\mathfrak{R}(h)\neq 1$. The first of these implies that $\deg\mathfrak{R}\leqq n$. Hence we conclude that H is a subdirect product of linear groups of finite degree. By Proposition 7.3 of [**3**], each such linear group being finitely generated is the subdirect product of finite groups. Hence the result follows.

Now only a finite number of group elements of H occur in the expressions for the β_i and the $[g_{i,1}, \cdots, g_{i,2n}]$. Let these be $h_1, \cdots, h_s$. Then we can write $\beta_i=\sum c_{ij}h_j$ and $[g_{i,1}, \cdots, g_{i,2n}]=\sum d_{ij}h_j$ with $c_{ij}, d_{ij}\in\mathbf{C}$. Let $\mathfrak{Z}$ be the finite set containing (1) $\mathfrak{K}_n(H)$, (2) all elements of the form $h_jh_k^{-1}$, and (3) all elements of the form $h_jh_{j'}^{-1}h_{k'}h_k^{-1}$. By the above there exists a normal subgroup N of H of finite index with $N\cap\mathfrak{Z}=\{1\}$. We show now that with this N, (iv) and (v) of Theorem 4 follow.

Since $N\cap\mathfrak{Z}=\{1\}$ and $h_jh_k^{-1}\in\mathfrak{Z}$ it follows that under the homomorphism $H\rightarrow\overline{H}=H/N$, that $\bar{h}_j$ (the image of h_j) is not equal to $\bar{h}_k$. With this we see that $\bar{\beta}_i\neq 0$ and that $[\bar{g}_{i1}, \cdots, \bar{g}_{i2n}]\neq 0$. The latter implies in addition that $b(\overline{H})>n$. Since any irreducible representation of $\mathbf{C}[\overline{H}]$ can be viewed as one of $\mathbf{C}[H]$ we have $N\mathfrak{K}_n(H)/N \subseteq \mathfrak{K}_n(\overline{H})$. But $N\mathfrak{K}_n(H)/N\simeq\mathfrak{K}_n(H)/(N\cap\mathfrak{K}_n(H))\simeq\mathfrak{K}_n(H)$ since $N\cap\mathfrak{K}_n(H)=1$. Hence $\mathfrak{K}_n(H)=\mathfrak{K}_n(G)$ is contained isomorphically in $\mathfrak{K}_n(\overline{H})$. We need only show that the isomorphism is onto. Let $\bar{g}\in\mathfrak{K}_n(\overline{H})$ with g an inverse image of $\bar{g}$. It suffices to show that $g\in N\mathfrak{K}_n(H)$.

Since $\bar{g} \in \mathfrak{K}_n(\overline{H})$ we have for all i

$$\bar{g}[\bar{g}_{i1}, \cdots, \bar{g}_{i2n}] = [\bar{g}_{i1}, \cdots, \bar{g}_{i2n}].$$

In $\mathbf{C}[G]$ this yields clearly

$$g(\sum d_{ij}h_j) = \sum d_{ij}n_{ij}h_j$$

with $n_{ij} \in N$. This follows since all the $\bar{h}_j$ are distinct. We show now that all the n_{ij} are equal. Consider one such element $n_{ij}h_j$. This comes from a term $gh_{j'}$ on the left of the above equation. Thus $gh_{j'} = n_{ij}h_j$ or $g = n_{ij}h_jh_{j'}^{-1}$. Replacing i by i', j by k, and j' by k' we also have $g = n_{i'k}h_kh_{k'}^{-1}$. Thus

$$n_{ij}^{-1} n_{i'k} = h_j h_{j'}^{-1} h_{k'} h_k^{-1} \in N \cap \mathfrak{Z} = \{1\}$$

so $n_{ij} = n_{i'k}$. Let their common value be n_{11}. Then for all i

$$n_{11}^{-1} g[g_{i1}, \cdots, g_{i2n}] = [g_{i1}, \cdots, g_{i2n}].$$

By the choice of the g_{ij} this implies that $n_{11}^{-1}g \in \mathfrak{K}_n(G) = \mathfrak{K}_n(H)$ and $g \in N\mathfrak{K}_n(H)$. This completes the proof.

As an application of the above result we prove

THEOREM 5. *Let* $\mathfrak{K}_n(G) > 1$. *Then* $b(G) \leqq n^2$.

PROOF. Suppose by way of contradiction that $b(G) > n^2 = m$. Then we can find group elements $g_1, \cdots, g_{2m}$ such that $[g_1, \cdots, g_{2m}] \neq 0$. Set $\beta_i = g_i$ and $\beta_{2m+1} = [g_1, \cdots, g_{2m}]$. Applying Theorem 4 we obtain a finite group $\overline{H}$ with $\mathfrak{K}_n(\overline{H}) > 1$ and containing elements $\bar{g}_1, \cdots, \bar{g}_{2m}$ with $[\bar{g}_1, \cdots, \bar{g}_{2m}] \neq 0$. Hence $b(\overline{H}) > n^2$.

Let $\bar{h} \in \mathfrak{K}_n(\overline{H})$ with $\bar{h} \neq 1$. Let θ be an irreducible complex character of $\overline{H}$ of degree $> n^2$. This exists since $b(\overline{H}) > n^2$. Clearly $\bar{h}$ is in the kernel of θ, that is $\theta(\bar{h}) = \theta(1) = \deg \theta$. Since $\mathbf{C}[\overline{H}]$ is semisimple we can find an irreducible character χ of $\overline{H}$ with $\bar{h}$ not in the kernel of χ. Let $\theta\chi = \sum a_i \chi_i$ where the χ_i are irreducible. Since $\bar{h} \notin$ kernel of $\theta\chi$ there exists a χ_i, say χ_1 with $\bar{h} \notin$ kernel χ_1. Now χ_1 is a constituent of $\theta\chi$ so

$$1 \leqq [\theta\chi, \chi_1] = [\theta, \bar{\chi}\chi_1]$$

where $[\ ,\ ]$ denotes the usual inner product of characters. Hence θ is a constituent of $\bar{\chi}\chi_1$. This yields

$$n^2 < \deg \theta \leqq \deg \bar{\chi}\chi_1 = (\deg \chi)(\deg \chi_1).$$

Clearly at least one of $\deg \chi$ or $\deg \chi_1$ is $> n$ and this is the required contradiction.

By Theorem F of [3] groups G with $b(G) \leqq n^2$ all have abelian subgroups of index $\leqq J(2n^2)$, where J is the function associated with Jordan's theorem on finite linear groups. Thus $\mathfrak{K}_n(G) > 1$ is a rather restrictive condition for a group to satisfy. We discuss now a method of constructing a class of groups G with $b(G) > n$ and $\mathfrak{K}_n(G) > 1$.

Let p be a fixed prime and let $e \geqq p$. Suppose we have e groups H_i each having a central subgroup $Z_i = \langle z_i \rangle$ of order p. Set

$$\begin{aligned} a_i &= \text{minimal degree of irreducible character } \theta_i \text{ of } H_i \text{ with } Z_i \not\subseteq \ker \theta_i; \\ b_i &= b(H_i); \\ c_i &= b(H_i/Z_i). \end{aligned}$$

We suppose further that for all i

$$(b_i/c_i) > \prod_{j=1}^{e} (b_j/a_j). \tag{1}$$

Let U be an abelian group of type (p, p) generated by $u, v \in U$. We define a homomorphism of

$$Z_1 \times Z_2 \times \cdots \times Z_e \to U$$

by $z_i \to uv^i$. This is clearly onto. Let the kernel be N. Then N is a central and hence normal subgroup of $H = H_1 \times H_2 \times \cdots \times H_e$. Set $G = H/N$ so that $G \supseteq U$, a central subgroup of type (p, p). Set $n+1 = \prod a_i$. We show that $v \in \mathfrak{K}_n(G)$ and that $b(G) > n$.

Let θ be an irreducible character of G. Then since U is central $\theta | U = (\deg \theta)\lambda$ where λ is a linear character of U. Hence some subgroup of order p of U is the kernel of θ. The subgroups of U are of course $\langle v \rangle$ and $\langle uv^i \rangle$ for $i = 1, 2, \cdots, p$. Since G is a homomorphic image of H, θ can be viewed as a character of H. In H we write $\theta = \theta_1\theta_2 \cdots \theta_e$ where θ_i is an irreducible character of H_i.

Suppose first that $\langle v \rangle \not\subseteq \ker \theta$. Then for some $i = 1, 2, \cdots, p$ we have $\langle uv^i \rangle \subseteq \ker \theta$. Then clearly in H, $z_i \in \ker \theta_i$. Hence $\deg \theta_i \leqq c_i$ and of course for $j \neq i$, $\deg \theta_j \leqq b_j$. Thus

$$\deg \theta \leqq \left(\prod b_j\right)(c_i/b_i) < \prod a_j = n + 1.$$

Hence $\deg \theta \leqq n$ and $v \in \mathfrak{K}_n(G)$. Now choose θ to be a character of $G/\langle v \rangle$ which is faithful on cyclic $U/\langle v \rangle$. Viewed in H we see that for all i, $z_i \notin \ker \theta_i$. Hence $\deg \theta_i \geqq a_i$ and so $\deg \theta \geqq \prod a_i = n+1$. Therefore $b(G) > n$ and the result follows.

Using the above we can easily construct some examples.

EXAMPLE 6. Let each H_i be a nonabelian group of order p^3. Then

$a_i=p$, $b_i=p$ and $c_i=1$ and so equation (1) is satisfied. This yields groups G nilpotent of class 2.

Indecomposable nonnilpotent groups with nontrivial kernels can be obtained as follows.

EXAMPLE 7. Let P denote the quaternion group of order 8 if $p=2$ or the nonabelian group of order p^3 and period p if p is odd. Let Z be the center of P. We have $|Z|=p$. Let A denote the group of automorphisms of P which centralize Z. A is easily seen to be isomorphic to the Symplectic group $S_{p_2}(p)$ whose order is $p(p+1)(p-1)$. Let $\alpha\in A$ be of prime order $q\neq p$ and let H be the semidirect product of P by the cyclic group $\langle\alpha\rangle$. Thus P is normal in H with index q and Z is central in H.

Let χ be an irreducible character of H. By Proposition 1.2 of [2], either $\chi|P$ is irreducible or $\chi|P$ is the sum of q conjugate characters under the action of $\langle\alpha\rangle$. In the first case $\deg\chi=1$ or p. In the second case let ϕ be an irreducible constituent of $\chi|P$. If $\deg\phi=p$ then ϕ vanishes off Z. Since α centralizes Z, $\phi^\alpha=\phi$, a contradiction. Hence $\deg\phi=1$ and $\deg\chi=q$. Thus H has characters of degree 1, p and q only. Moreover if $H=H_i$ we have easily $a_i=p$, $b_i=\max(p, q)$ and $c_i=q$.

If $p=2$ choose $q=3$. Then $b(H)=3$ and $Z\subseteq\mathfrak{K}_2(H)$. If $p>2$ then choose q to divide $p(p+1)(p-1)$ so $p>q$. Hence in this case equation (1) is satisfied and the group G constructed has the required property.

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