EXISTENCE OF INVARIANT MEASURES FOR MARKOV PROCESSES. II[1]

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The purpose of this note is to improve the results obtained in [2] by relaxing the assumptions on the processes under consideration. Thus we impose weaker topological structure on the states space and, what is more important, the continuity of $P(x, A)$, when A is open, is *not* assumed.

1. **Notation.** We shall use the notation of [1] for topological and measure theoretic concepts.

Let X be a normal topological space. Let $P(x, A)$ be the transition probabilities of a Markov Process:

1.1. *For a fixed $x \in X$ the set function $P(x, \cdot)$ is a measure, on the Borel sets, of total measure one.*

1.2. *For a fixed Borel set A, the function $P(\cdot, A)$ is Borel measurable.*

By a measure we shall mean a countably additive positive measure, unless otherwise stated. Let us denote by **r b a** the set of regular bounded finitely additive signed measures on X and by **r c a** those elements of **r b a** which are countably additive. The transition probabilities induce an operator on the bounded measurable functions by

1.3. $(Pf)(x) = \int f(y) P(x, dy)$.

Also if μ is a bounded finitely additive signed measure one defines

1.4. $(\mu P)(A) = \int P(x, A) \mu(dx)$.

It is well known that

1.5. $\int (Pf)(x) \mu(dx) = \int f(x) (\mu P)(dx)$

and that μP is countably additive if μ is.

Throughout the paper we assume:

1.6. *If $f \in C(X)$ then $Pf \in C(X)$*, where $C(X)$ denotes the continuous functions. Also:

1.7. *If $\mu \in$* **r c a** *then $\mu P \in$* **r c a**.

These two conditions are always satisfied under the assumptions of [2]: under the assumptions of [2] every countably additive measure is regular. Another example is given by $(Pf)(x) = f(\phi(x))$ where ϕ is a homeomorphism of X onto X.

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2. **Invariant measures.** Let P^* be the adjoint operators of P on $C(X)$. Thus by Theorem IV.6.2 of [1] P^* is defined on **r b a** and from 1.5 and 1.7 follows that:

2.1. *If* $\mu \in$ **r c a** *then* $P^*\mu = \mu P$.

Let μ be a positive finitely additive measure. Then $\mu = \mu_1 + \mu_2$ where μ_1 is countably additive and μ_2 is purely finitely additive i.e.: if $\mu_2 \geqq \lambda \geqq 0$ and λ is countably additive then $\lambda = 0$. Also $\mu_1 \geqq 0$, $\mu_2 \geqq 0$. This decomposition and its uniqueness are proved in [3, p. 52]. Clearly if μ is regular so are μ_1 and μ_2.

LEMMA 1. *Let* $\mu \in$ **r b a** *and* $\mu \geqq 0$. *If* $P^*\mu = \mu$ *then* $P^*\mu_1 = \mu_1$.

PROOF. Since $\mu_1 + \mu_2 = P^*\mu_1 + P^*\mu_2$ and $P^*\mu_1$ is countably additive it follows that $\mu_1 \geqq P^*\mu_1$: let $P^*\mu_2 = \sigma_1 + \sigma_2$ where σ_1 is countably additive and σ_2 purely finitely additive then $\mu_1 = P^*\mu_1 + \sigma_1$. But

$$\mu_1(X) = \int d\mu_1 = \int P1 d\mu_1 = (P^*\mu_1)(X),$$

where 1 is the function identically to one and by 1.1 $P1 = 1$. Thus $0 \leqq (\mu_1 - P^*\mu_1)(A) \leqq (\mu_1 - P^*\mu_1)(X) = 0$.

For any $0 \leqq \mu \in$ **r b a** put

$$\mu_n = \frac{\mu + P^*\mu + \cdots + P^{*n-1}\mu}{n}. \tag{2.2}$$

THEOREM 1. *Let A be a fixed compact set. Then either*

(a) *for every* $0 \leqq \mu \in$ **r b a** $\lim \mu_n(A) = 0$, *or*

(b) *there exists a measure* $0 \leqq \mu \in$ **r c a** *with* $\mu = \mu P$ *and* $\mu(A) \neq 0$.

PROOF. Let $0 \leqq \mu \in$ **r c a** be such that $\mu_{n_i}(A) \geqq \delta > 0$ for a subsequence, n_i, of the integers. Let μ be a weak star limit of μ_{n_i}. Clearly $P^*\mu = \mu$. If B is any open set containing A let $f \in C(X)$ satisfy $0 \leqq f \leqq 1$, $f(X - B) = 0$, $f(A) = 1$. Then

$$\mu(B) \geqq \int f\, d\mu = \lim \int f\, d\mu_{n_i} \geqq \delta.$$

Since the measure μ is regular also $\mu(A) \geqq \delta$. Finally let $\mu = \mu_1 + \mu_2$ be the decomposition of Lemma 1. The theorem will be proved if we show that $\mu_2(A) = 0$ since then μ_1 will satisfy (b). But the restriction of μ_2 to A is countably additive, by Theorem III.5.13 of [1] and thus is zero.

Given an invariant measure $0 \leqq \mu \in$ **r c a** if $\int f d\mu = 0$ where $0 \leqq \in C(X)$, then $\int P f d\mu = 0$ too, hence $\int (\sum_{n=1}^{\infty} P^n f) d\mu = 0$. If P is such that whenever $0 \leqq f \in C(X)$ and $f \neq 0$ $\sum P^n f > 0$ then μ never vanishes

on open sets, or the kernel of μ is all of X.

Let $K(\mu)$ be the kernel of $0 \leqq \mu \in \mathbf{r\,c\,a}$ i.e.: if $x \in K(\mu)$ and N is a neighborhood of x then $\mu(N) \neq 0$.

THEOREM 2. *Let* $0 \leqq \mu \in \mathbf{r\,c\,a}$ *be invariant. Then*

$$P^n(x, K(\mu)) = 1, \qquad x \in K(\mu), \qquad n = 1, 2, \cdots. \tag{2.3}$$

PROOF. Let us show (2.3) for $n=1$. Take a fixed $x \in K(\mu)$. The measure $P(x, \cdot) = \delta_x P$ is regular by 1.7. Thus it is enough to show that if A is a closed set disjoint to $K(\mu)$ then $P(x, A) = 0$. Let $f \in C(X)$ be such that $0 \leqq f \leqq 1$, $f(K(\mu)) = 0$ and $f(A) = 1$. Then

$$0 = \int f\, d\mu = \int (Pf)\, d\mu.$$

Thus $Pf = 0$ a.e. and since Pf is continuous and $x \in K(\mu)$ $(Pf)(x) = 0$. Finally $P(x, A) \leqq (Pf)(x)$ since $f(A) = 1$.

Following [2] let us define:

DEFINITION. *A set* $A \subset X$ *is called self-contained if* $P(x, A) = 1$ *for all* $x \in A$.

Also put for a self contained set A

$$A^n = \{x : P^n(x, A) > 0\}, \qquad A^* = \bigcup_{n=1}^{\infty} A^n - A. \tag{2.4}$$

Then if μ is an invariant measure

$$\mu(A^*) = 0 \tag{2.5}$$

and also if A is self contained so is $X - A^* - A$.

These facts are proved in Theorem 4 and Lemma 5 of [2], respectively, and the proof is valid in our case too.

It should be noted that even when A is closed $A \cup A^*$ does not have to be open: consider the identity transformation. Thus we can not continue to prove results obtained in Theorem 6 and 7 of [2]. Clearly the proof of Theorem 9 fails in our case.

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