A PROPERTY OF THE REAL NOT REGULAR FUNCTIONS C^∞

HANS HORNICH

Let f be $\in C^\infty$ on the interval $I=[0, \mathfrak{L}]$, $\mathfrak{L}>0$ and

$$\limsup \left(\left|f^{(n)}(0)\right|/n!\right)^{1/n} < +\infty. \tag{1}$$

If f is not regular for $x=0$, i.e. if f cannot be expanded into a power series in x on $[0, \delta]$ for any $\delta>0$, then there exists a $n(K)$ for every $K>0$ such that f cannot be a solution of a linear differential equation on I

$$y^{(n)} + \sum_{i=0}^{n-1} \sigma_i y^{(i)} = \varphi \tag{2}$$

with any $\mathfrak{L}$-integrable functions σ_i, φ on I with

$$\left|\sigma_i(x)\right| < K, \qquad \left|\varphi(x)\right| < K$$

for $x\in I$ and $n>n(K)$.

PROOF. If f satisfies a differential equation (2) then

$$\bar{f}(x) = f(x) - \sum_{i=0}^{n-1} f^{(i)}(0)x^i/i! = f(x) - P(x) \tag{3}$$

satisfies a differential equation

$$\bar{y}^{(n)} + \sum_{i=0}^{n-1} \sigma_i \bar{y}^{(i)} = \varphi - \sum_{i=0}^{n-1} \sigma_i P^{(i)} = \bar{\Phi} \tag{4}$$

with

$$\bar{f}(0) = \bar{f}'(0) = \cdots = \bar{f}^{(n-1)}(0) = 0. \tag{5}$$

Now

$$\sum_{i=0}^{n-1} \sigma_i P^{(i)} = \sum_{i=0}^{n-1} \sigma_i \sum_{j=i}^{n-1} f^{(j)}(0)x^{j-i}/(j-i)! \tag{6}$$

and with

$$\max_i \sup_{x\in I} \left|\sigma_i(x)\right| = S$$

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we get the inequality

$$\left|\sum_{i=0}^{n-1}\sigma_i P^{(i)}\right| \leqq S\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\left|f^{(j)}(0)\right| x^{j-i}/(j-i)!. \tag{7}$$

Because of (1) we have for an appropriate $A > 1$

$$\left|f^{(j)}(0)\right| \leqq A^j j! \tag{8}$$

so

$$\begin{aligned}\left|\sum_{i=0}^{n-1}\sigma_i P^{(i)}\right| &\leqq S\sum_{i=0}^{n-1}\sum_{j=i}^{n-1} A^j j(j-1)\cdots(j-i+1)x^{j-i} \\ &\leqq SA^{n-1}n^2(n-1)!\end{aligned} \tag{9}$$

for $x < 1$.

We prove the following

LEMMA. *For the solution $\bar{y}$ of a differential equation*

$$\bar{y}^{(n)} + \sum_{i=0}^{n-1}\sigma_i\bar{y}^{(i)} = \bar{\Phi} \tag{10}$$

with

$$\bar{y}(0) = \bar{y}'(0) = \cdots = \bar{y}^{(n-1)}(0) = 0 \tag{11}$$

the following inequality is valid:

$$\left|\bar{y}(x)\right| \leqq \phi\mathfrak{L}x^{n-1}\Big/\left(1 - \sum_{i=0}^{n-1}S_i\mathfrak{L}^{n-i}/(n-i)!\right)(n-1)! \tag{12}$$

on $I = [0, \mathfrak{L}]$ with

$$S_i = \sup_{x\in I}\left|\sigma_i(x)\right|, \qquad \phi = \sup_{x\in I}\left|\bar{\Phi}(x)\right|$$

where $\mathfrak{L}$ is so small that

$$\sum_{i=0}^{n-1}S_i\mathfrak{L}^{n-i}/(n-i)! < 1. \tag{13}$$

We put

$$\sup_{x\in I}\left|\bar{y}^{(n-1)}(x)\right| = Y;$$

then it follows that

$$\left|\bar{y}^{(i)}(x)\right| \leqq Yx^{n-1-i}/(n-1-i)! \tag{14}$$

for $i=0, \cdots, n-1$.

Now from (10) we get

$$\bar{y}^{(n-1)}(x) = \int_0^x \left[\bar{\Phi}(t) - \sum_{i=0}^{n-1} \sigma_i(t)\bar{y}^{(i)}(t)\right] dt$$

and because of (14)

$$|\bar{y}^{(n-1)}(x)| \leqq \phi x + \sum_{i=0}^{n-1} S_i Y x^{n-i}/(n-i)! \tag{15}$$

and

$$Y \leqq \phi \mathfrak{L} + \sum_{i=0}^{n-1} S_i Y \mathfrak{L}^{n-i}/(n-i)! \tag{16}$$

or

$$|\bar{y}^{(n-1)}(x)| \leqq Y \leqq \phi\mathfrak{L} \bigg/ \left(1 - \sum_{i=0}^{n-1} S_i \mathfrak{L}^{n-i}/(n-i)!\right) \tag{17}$$

and finally by integration (12).

We apply this Lemma to (4); if $\mathfrak{L}$ is supposed to be so small that

$$\mathfrak{L} < 1, \ A\mathfrak{L} < 1 \quad \text{and} \quad S(e^{\mathfrak{L}} - 1) < 1, \tag{18}$$

then because of (9)

$$|\bar{y}(x)| \leqq [K + Sn^2A^{n-1}(n-1)!]\mathfrak{L}x^{n-1}/(n-1)![1 - (e^{\mathfrak{L}} - 1)S]. \tag{19}$$

Now f is not regular for $x=0$; therefore, for every natural number m we have

$$\sup_{x\in I} \left| f(x) - \sum_{i=0}^{m-1} f^{(i)}(0)x^i/i! \right| = h_m > 0 \tag{20}$$

and

$$\inf_m h_m = h > 0.$$

There exists, for every m, an x_m on I with

$$\left| f(x_m) - \sum_{i=0}^{m-1} f^{(i)}(0)x_m^i/i! \right| \geqq h. \tag{21}$$

The right side of (19) becomes arbitrarily small for $n \to \infty$; therefore, an $n(K)$ exists for a given K such that

$$|\bar{y}(x)| < h \tag{22}$$

holds for $x \in I$, that is, for all functions σ_i, with

$$S \leqq K, \qquad |\varphi(x)| < K$$

and $n > n(K)$. The inequalities (19) and (21) are both valid only if $m < n(K)$ holds in

$$\bar{y}(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0)x^i/i!.$$

We add some consequences. Let f be $\in C^\infty$ on $I = [0, \mathfrak{L}]$. If

[i] for every point $x \in I$ there is

$$\limsup (|f^{(n)}(x)|/n!)^{1/n} < +\infty \tag{23}$$

and

[ii] if there exists a sequence of linear differential equations of increasing order with uniformly bounded coefficients such that f satisfies every differential equation of this sequence, then f is regular in every point of I.

This is an immediate consequence of our theorem.

The validity of (23) in every point of I is a necessary condition for the regularity of f.

If f satisfies a linear differential equation with uniform bounded coefficients for every order $n > n_0$, then (23) is valid for every point of I and f is regular on I.

It is necessary to give the proof for the first part only; if (23) is not valid for a point x_0 on I, then there exists a subsequence $\{n_K\}$ of natural numbers such that $|f^{(n_K)}(x_0)|$ are increasing in such a way that the equation (2) cannot be true for the point x_0 for sufficiently large n_K.

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