THE EXISTENCE OF PROPER SOLUTIONS OF A SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

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We shall consider the equation

$$\begin{aligned} x'' &= f(t, x, x') : f(t, x, y) \text{ continuous on } D \\ &= \{(t, x, y) : 0 \leqq t < +\infty, -\infty < x, y < +\infty\}. \end{aligned} \tag{1}$$

We further assume: (i) $f(t, x, y)$ is such that the solutions of (1) are uniquely determined by initial conditions; (ii) $xf(t, x, 0) > 0$ if $t > 0$, $x \neq 0$; (iii) $f(t, 0, 0) = 0$ for all $t \geqq 0$; (iv) if $u > v$, then $f(t, u, y) \geqq f(t, v, y)$, if $r > s$, then $f(t, x, r) \geqq f(t, x, s)$.

From (ii), a positive solution of (1) has no relative maxima, a negative solution has no relative minima. If $x(0) > 0$ and $x(c) = 0$, $x'(c) < 0$ for some $c > 0$, then $x(t) < 0$ for all $t > c$. The behavior of positive solutions and negative solutions is similar, so we shall consider only the former. If we totally disregard nonpositive solutions, we can weaken (ii) to $f(t, x, 0) > 0$ for $x, t > 0$.

A solution, $x(t)$, of (1) is *proper* if $x(t)$ exists and is positive for all $t \geqq 0$. If $f(t, x, y)$ satisfies (i)–(iv), we shall show that given $A > 0$ there exists a unique proper solution of (1), $x(t)$, such that $x(0) = A$ and $x'(t) < 0$ for all $t \geqq 0$. We shall use the topological method of T. Ważewski ([**2**] or [**1**, pp. 179–182]). This approach generalizes a result (and simplifies the proof) of P. K. Wong [**3**, Theorem 1.1].

Let $u(t)$ and $v(t)$ be two solutions of (1). If $u(t)$ and $v(t)$ are defined for $0 \leqq t < a \leqq +\infty$, then

$$\begin{aligned} u'(t) - v'(t) &= u'(0) - v'(0) \\ &\quad + \int_0^t [f(r, u(r), u'(r)) - f(r, v(r), v'(r))] dr. \end{aligned} \tag{2}$$

If $u(t)$ and $v(t)$ are defined for $0 \leqq t \leqq b < +\infty$, then

$$\begin{aligned} u(t) - v(t) &= u(0) - v(0) + [u'(0) - v'(0)]t \\ &\quad + \int_0^t \int_0^s f(r, u(r), u'(r)) - f(r, v(r), v'(r))] \, dr \, ds. \end{aligned} \tag{3}$$

THEOREM 1. *If $u(t)$ and $v(t)$ are two proper solutions of* (1) *with $u(0) = v(0) = A > 0$, $u'(\infty) = v'(\infty) = 0$, then $u(t) = v(t)$ for all $t \geqq 0$.*

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PROOF. Let $u(t)$ and $v(t)$ satisfy the hypotheses and assume $v'(0) < u'(0)$. Suppose there exists a $c > 0$ such that $v(t) < u(t)$ and $v'(t) < u'(t)$ for $0 < t < c$ and either $v(c) = u(c)$ or $v'(c) = u'(c)$. From (2) and (iv), $u'(c) - v'(c) \geqq u'(0) - v'(0) > 0$; and from (3) and (iv), $u(c) - v(c) \geqq [u'(0) - v'(0)]c > 0$. Therefore, $v(t) < u(t)$ and $v'(t) < u'(t)$ for $0 \leqq t < +\infty$ and $0 = u'(\infty) - v'(\infty) \geqq [u'(0) - v'(0)] > 0$, a contradiction. Thus, $v'(0) \geqq u'(0)$—and likewise $u'(0) \geqq v'(0)$—so $u(t) = v(t)$ for all $t \geqq 0$.

LEMMA. *Given $A > 0$, $c > 0$ there exists a $d(A, c) > 0$ such that if $x(t)$ is a solution of* (1) *with $0 < x(0) \leqq A$ and $x'(0) \leqq -d$, then $x(t) = 0$ for some $t \in (0, c)$.*

PROOF. Let $T = \{(t, x): 0 \leqq t \leqq c,\ 0 \leqq x \leqq A - t(A/c)\}$ and let H be the hypotenuse of T. We will show that solutions satisfying the hypotheses do not cross H for $0 \leqq t \leqq c$.

Let $M = \max\{f(t, x, 0): (t, x) \in T\}$, $M > 0$ by (ii), and let $d = (A/c) + Mc$. Let $x(t)$ be a solution of (1) with $0 < x(0) \leqq A$ and $x'(0) \leqq -d$. For t small, $x(t)$ lies in T and below H. If $x(t)$ strikes H for some $t \in (0, c]$, then there exists an $e \in (0, c)$ such that $x'(e) = -(A/c)$, and $x'(t) < -(A/c)$, $(t, x(t)) \in T$ for $0 \leqq t < e$. Then

$$x'(e) = x'(0) + \int_0^e f(s, x(s), x'(s))ds$$
$$\leqq -(A/c) - Mc + Me < -(A/c).$$

THEOREM 2. *Given $A > 0$ there exists exactly one proper solution, $x(t)$, of* (1) *such that $x(0) = A$ and $x'(t) < 0$ for all $t \geqq 0$.*

PROOF. Theorem 1 shows the uniqueness of such solutions.

Write (1) as a system

$$\begin{aligned} x' &= y, \\ y' &= f(t, x, y); \end{aligned} \tag{4}$$

we seek a solution, $(x(t), y(t))$, of (4) such that $x(0) = A > 0$, $y(0) < 0$, and $x(t) > 0$, $y(t) < 0$ for all $t \geqq 0$. We will now use the method and terminology of Ważewski. Let $T = \{(t, x, y): t \geqq 0,\ x > 0,\ y < 0\}$; $Q = \{(t, x, y): t \geqq 0, x > 0, y = 0\}$; and $R = \{(t, x, y): t \geqq 0, x = 0, y < 0\}$. For solutions of (4) with $t > 0$, the set of egress of T is $S = Q \cup R$ (by (i), the t-axis contains no points of egress). Every point in Q is a point of strict egress since $y' = f(t, x, 0) > 0$ and every point in R is a point of strict egress since $x' = y < 0$ on R. Thus, the set of egress equals the set of strict egress.

Now let $X=\{(t, x, y): t=0, x=A, -d\leqq y\leqq 0\}$ (where d is determined in the lemma); $Y=\{(t, x, y): t=0, 0\leqq x\leqq A, y=-d\}$; and $Z=X\cup Y$. Then $S\cap Z$ is a retract of S but not of Z and by Ważewski's theorem there exists a point $P=(0, a, b)\in Z$ such that the solution of (4) with $x(0)=a$, $y(0)=b$ remains in T for all $t\geqq 0$. And by the lemma, $P\notin Y$.

References

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