THE NUMBER OF ISOMORPHISM TYPES
OF FINITE ALGEBRAS[1]

MICHAEL A. HARRISON

1. Introduction. An *abstract algebra* is a system $\mathfrak{A} = \langle A, o_i \rangle_{i \in I}$ where each operation o_i is a single valued function which assigns to every l_i-tuple of elements of A, say $(x_1, \cdots, x_{l_i})$, a value $o_i(x_1, \cdots, x_{l_i}) \in A$; l_i is the *rank* of o_i for $i \in I$ and it is assumed that $l_i \geqq 1$ for all $i \in I$. The rank of o is denoted by $\rho(o)$; the assumption $\rho(o) \geqq 1$ will be relaxed later. $\mathfrak{A}$ is said to be *finite* if A is finite and I is finite. Assume that $\mathfrak{A} = \langle A, o_1, \cdots, o_k \rangle$ is an algebra in which each operation o_i is of rank l_i. The *similarity type* of $\mathfrak{A}$ is $l = (l_1, \cdots, l_k)$. Let $\mathfrak{A} = \langle A, o_1, \cdots, o_k \rangle$ and $\mathfrak{B} = \langle B, o_1', \cdots, o_m' \rangle$ be two algebras; $\mathfrak{A}$ is said to be *isomorphic* to $\mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ have the same similarity type and there is a one-to-one map ϕ from A onto B such that for any $(x_1, \cdots, x_{l_i}) \in A^{l_i}$,

$$\phi o_i(x_1, \cdots, x_{l_i}) = o_i(\phi(x_1), \cdots, \phi(x_{l_i})) \quad (i = 1, \cdots, k = m).$$

Since isomorphism is an equivalence relation, we speak of the *isomorphism type* of $\mathfrak{A}$ as the equivalence class containing $\mathfrak{A}$.

In this paper, the number of isomorphism types of finite algebras is derived.[2] In addition, both lower bounds and algorithms for determining the number of algebras with distinguished constants are obtained. Since finite automata can be considered as finite algebras in which all the operations are unary, the present results form a generalization of the results of [4].

2. The counting method. Let $\mathfrak{G}$ be a permutation group of order g and degree s acting on a set S. Two elements $s_1, s_2 \in S$ are called *equivalent* iff there is an element $\alpha \in \mathfrak{G}$ such that $s_1 = \alpha(s_2)$. The number of equivalence classes is given by a theorem due to Frobenius [1, p. 191].

THEOREM 1. *Let $\mathfrak{G}$ be a permutation group of order g and degree s acting on S. The number of equivalence classes induced by $\mathfrak{G}$ is*

$$\frac{1}{g} \sum_c n_c I(c),$$

where the sum is over all conjugate classes c, $n_c = |c|$, and $I(c)$ denotes the number of fixed points of S under any permutation in c.

It is useful to introduce a generating function for the cycle structure of $\mathfrak{G}$. Let $f_1, \cdots, f_s$ be s indeterminates and let α be any permutation of S having j_i cycles of length i for $i = 1, \cdots, s$. It is convenient to use the term *cycle structure* of α and to write this $(j_1, \cdots, j_s)$ or $f_1^{j_1} \cdots f_s^{j_s}$. Since every element in S is in exactly one cycle of the permutation α, we have

$$(1) \qquad \sum_{i=1}^{s} i j_i = s.$$

The *cycle index polynomial* of $\mathfrak{G}$ is defined to be

$$Z_{\mathfrak{G}}(f_1, \cdots, f_s) = \frac{1}{g} \sum_{(j)} g_{(j)} f_1^{j_1} \cdots f_s^{j_s},$$

where $g_{(j)} = g_{(j_1, \ldots, j_s)}$ is the number of elements of $\mathfrak{G}$ having cycle structure (j); the summation is over all non-negative integer solutions (j) of equation (1), i.e., over all partitions of s.

The applications in this paper will require only the special case when $\mathfrak{G} = \mathfrak{S}_n$, the symmetric group of degree n. It is well known that

$$Z_{\mathfrak{S}_n}(f_1, \cdots, f_n) = \frac{1}{n!} \sum_{(j)} \frac{n!}{\displaystyle\prod_{i=1}^{n} j_i! i^{j_i}} f_1^{j_1} \cdots f_n^{j_n}$$

where the summation is over all partitions of n.

In counting abstract algebras, it is necessary to obtain the cycle structure of a permutation induced on the cartesian product of two sets arising from permutations acting on each set individually. The following definition and theorem are due to Harary [3].

Let $\mathfrak{A}$ and $\mathfrak{B}$ be permutation groups of order m and n operating on disjoint object sets X and Y of cardinality a and b respectively. The *cartesian product* of $\mathfrak{A}$ and $\mathfrak{B}$, denoted by $\mathfrak{A} \times \mathfrak{B}$, is defined on $X \times Y$ as

$$(\alpha, \beta)(x, y) = (\alpha(x), \beta(y)).$$

It is important to be able to compute the cycle index of $\mathfrak{A} \times \mathfrak{B}$ from the cycle indices of $\mathfrak{A}$ and of $\mathfrak{B}$. This is accomplished by defining a cross operation on cycle index polynomials. The pertinent result of Harary is the following.

Theorem 2 (Harary). *If*

$$Z_{\mathfrak{A}} = \frac{1}{m} \sum_{(i)} a_{(i)} \prod_{r=1}^{a} g_r^{i_r}$$

and

$$Z_{\mathfrak{B}} = \frac{1}{n} \sum_{(j)} b_{(j)} \prod_{s=1}^{b} h_s^{j_s},$$

then

$$Z_{\mathfrak{A} \times \mathfrak{B}} = Z_{\mathfrak{A}} \times Z_{\mathfrak{B}} = \frac{1}{m} \frac{1}{n} \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} \prod_{r=1}^{a} \prod_{s=1}^{b} f_{\langle r,s \rangle}^{i_r j_s (r,s)},$$

where $\langle r, s \rangle$ is the least common multiple of r and s, while (r, s) is the greatest common divisor of r and s.

De Bruijn [2] generalized Pólya's theorem to determine the number of classes of functions under separate groups $\mathfrak{G}$ and $\mathfrak{H}$ acting independently on the domain and range. This result is also needed here.

Let $\mathfrak{G}$ be a permutation group of order g acting on a set D of cardinality s while $\mathfrak{H}$ is a permutation group of order h acting on a set R of cardinality r. Consider the class of functions from D into R and call two functions f_1 and f_2 *equivalent* if there exists an $\alpha \in \mathfrak{G}$ and a permutation $\beta \in \mathfrak{H}$ such that for every $d \in D$, $f_1(\alpha(d)) = \beta f_2(d)$.

Theorem 3 (De Bruijn). *The number of classes of functions $f: D \to R$ with a permutation group $\mathfrak{G}$ of a degree s and order g acting on D and a group $\mathfrak{H}$ of degree r and order h acting on R is given by*

$$Z_{\mathfrak{G}} \left(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_s} \right) Z_{\mathfrak{H}} (h_1, \cdots, h_r)$$

evaluated at $z_1 = \cdots = z_s = 0$, where $h_i = \exp \sum_{k=1}^{s/t} i z_{ki}$ for $i = 1, \cdots, r$.

Theorem 3 is generally applied in actual problems through the use of the following lemma:

Lemma 1. *A term $h_1^{j_1} \cdots h_r^{j_r}$ in $Z_{\mathfrak{H}}$ gives rise to*

$$Z_{\mathfrak{G}} \left(\sum_{t | 1} t j_t, \cdots, \sum_{t | s} t j_t \right).$$

The device used to generalize these classical results is to note that one may use Theorem 3 and Lemma 1 for individual terms of the

cycle indices; the result of such computations is always the number of fixed points of the permutations. Cf. [4] for details.

3. **The main results.** We begin with some special cases which are of independent interest and because the final general formula is so complex.

THEOREM 4. *Let $\mathfrak{A} = \langle A, + \rangle$ be an algebra where $|A| = n$ and $+$ is a binary operation on A, i.e. $\mathfrak{A}$ is a groupoid. The number of isomorphism types of $\mathfrak{A}$ is*

$$\frac{1}{n!} \sum_{(j)} \frac{n!}{\prod_{i=1}^{n} j_i! i^{j_i}} \prod_{i_1=1}^{n} \prod_{i_2=1}^{n} \left(\sum_{d | \langle i_1, i_2 \rangle} d j_d \right)^{j_{i_1} j_{i_2} (i_1, i_2)},$$

where the summation is over all partitions of n; $\langle i_1, i_2 \rangle [(i_1, i_2)]$ denotes the least common multiple [greatest common divisor] of i_1 and i_2.

PROOF. An isomorphism ϕ of $\mathfrak{A}$ onto $\mathfrak{B}$ is a one-to-one map, hence a permutation of A. Suppose ϕ has cycle structure $f_1^{j_1} \cdots f_n^{j_n}$. Since $\phi(a+b) = \phi(a) + \phi(b)$, we determine the cycle structure that ϕ induces on A^2. This is given by

$$(f_1^{j_1} \cdots f_n^{j_n}) \times (f_1^{j_1} \cdots f_n^{j_n}) = \prod_{i_1=1}^{n} \prod_{i_2=1}^{n} f_{\langle i_1, i_2 \rangle}^{j_{i_1} j_{i_2} (i_1, i_2)}.$$

Applying Lemma 1, we have that the permutation ϕ leaves invariant

$$\prod_{i_1=1}^{n} \prod_{i_2=1}^{n} \left(\sum_{d | \langle i_1, i_2 \rangle} d j_d \right)^{j_{i_1} j_{i_2} (i_1, i_2)}$$

functions. Summing over all partitions of n and applying Theorem 3, we have the theorem.

The next result is for abstract algebras with a single l-ary function.

THEOREM 5. *Let $\mathfrak{A} = \langle A, o \rangle$ be an algebra where $|A| = n$ and $o: A^l \rightarrow A$. The number of isomorphism types of such algebras is*

$$\frac{1}{n} \sum_{(j)} \frac{n!}{\prod_{i=1}^{n} j_i! i^{j_i}} \prod_{i_1=1}^{n} \cdots \prod_{i_l=1}^{n} \left(\sum_{d | \langle i_1, \cdots, i_l \rangle} d j_d \right)^{j_{i_1} \cdots j_{i_l} (i_1, \cdots, i_l)}.$$

PROOF. Clearly[3]

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[3] We use $a^{\times k}$ as an abbreviation for $a \times \overbrace{\cdots}^{k} \times a$ to avoid confusion with ordinary exponents.

$$(f_1^{j_1} \cdots f_n^{j_n})^{\times l} = \prod_{i_1=1}^{n} \cdots \prod_{i_l=1}^{n} f_{\langle i_1, \cdots, i_l \rangle}^{j_{i_1} \cdots j_{i_l}(i_1, \cdots, i_l)}$$

The result follows from Lemma 1.

The next case is when the algebras has k operations all of rank l.

THEOREM 6. *Let* $\mathfrak{A} = \langle A, o_1, \cdots, o_k \rangle$ *be an algebra where* $|A| = n$ *and* $o_i: A^l \to A$ *for each* $i = 1, \cdots, k$. *The number of isomorphism types is*

$$\frac{1}{n!} \sum_{(j)} \frac{n!}{\prod_{i=1}^{n} j_i! i^{j_i}} \prod_{i_1=1}^{n} \cdots \prod_{i_l=1}^{n} \left(\sum_{d | \langle i_1, \cdots, i_l \rangle} d j_d \right)^{k j_{i_1} \cdots j_{i_l}(i_1, \cdots, i_l)} .$$

PROOF. The result follows from the previous theorem upon noting that each of the k operations must be considered independently.

Lastly, we consider the general case of k operations where each o_i has rank l_i. No loss of generality results from permuting the operations so as to keep operations of the same rank adjacent.

THEOREM 7. *Let* $\mathfrak{A} = \langle A, o_1, \cdots, o_k \rangle$ *where each* $o_i: A^{l_i} \to A$ *and* $|A| = n$; *assume that there are* m_r *operations of rank* r *for* $r = 1, \cdots,$ $\max_i (\rho(o_i)) = q$. *The number of isomorphism types of such algebras is*

$$\frac{1}{n!} \sum_{(j)} \frac{n!}{\prod_{i=1}^{n} j_i! i^{j_i}} \prod_{r=1}^{q} \left(\prod_{i_1=1}^{n} \cdots \prod_{i_{l_r}=1}^{n} \sum_{d | \langle i_1, \cdots, i_{l_r} \rangle} d j_d \right)^{m_{l_r} j_{i_1} \cdots j_{i_{l_r}}(i_1 \cdots i_{l_r})} .$$

4. Further results. It is often convenient to consider algebras with distinguished constants, e.g., $\langle R, +, \cdot, o, 1 \rangle$. We identify constants with functions of rank 0. The number of isomorphism types of such algebras is obtained by using the same trick employed in [4] to count automata with fixed initial states.[4]

THEOREM 8. *Let* $\mathfrak{A} = \langle A, o_1, \cdots, o_k, c_1, \cdots, c_m \rangle$ *be an abstract algebra with* $|A| = n \geq m$. *Let* $F_{\mathfrak{A}'}(n)$ *be the formula for the number of isomorphism types of algebras* $\mathfrak{A}' = \langle A, o_1, \cdots, o_k \rangle$ *with* $|A| = n$ *as given by Theorem 7. The number of isomorphism types of algebras* $\mathfrak{A} = \langle A, o_1, \cdots, o_k, c_1, \cdots, c_m \rangle$ *is given by*

$$S_{j_1+m}^{j_1} F_{\mathfrak{A}'}(n - m)$$

where the formula means that j_1 *is replaced by* $j_1 + m$ *throughout* $F_{\mathfrak{A}'}(n-m)$.

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[4] I believe that this device is originally due to Frank Harary who used it to count nonisomorphic rooted graphs.

TABLE 1. The number of types of algebras with one binary operation.

n	Number of Types
1	1
2	10
3	3,330
4	178,981,952

TABLE 2. The number of types of algebras with two binary operations.

n	Number of Types
1	1
2	136
3	64,573,605

PROOF. Under any isomorphism ϕ of $\mathfrak{A} = \langle A, o_i, \cdots, o_k, c_1, \cdots, c_m \rangle$ onto $\mathfrak{B} = \langle B, o_1', \cdots, o_1', c_1' \cdots, c_m' \rangle$ we must have $\phi(c_i) = c_i'$ for $i = 1, \cdots, m$. The $(n - m)$ remaining elements may be permuted by ϕ as before while the constants remain fixed.

Since the formulas of §3 are complex, it would be desirable to have simple estimates for the desired numbers. One immediately obtains the following lower bound in the general case,

$$\frac{1}{n!} \prod_{r=1}^{q} n^{m_{l_r} n^{l_r}}$$

by just taking the first term of the formula. In [4] and hence for unary algebras, the lower bound has also been shown to be the asymptotic estimate as n increases without limit. This simple result does not generalize to the present case. It would be desirable to obtain asymptotic estimates but this task falls outside the context of the present paper.

5. Numerical results. We present some numerical results in the most interesting cases. Since the number of types grows so rapidly, only very small values of n are considered. Note that the results of Table 4 as compared with Table 3 confirm that, with designated constants, the number of isomorphism types increases since there is less freedom in constructing isomorphisms.

TABLE 3. The number of types of algebras with one ternary operation.

n	Number of Types
1	1
2	130
3	1,270,932,917,454

TABLE 4. The number of types of algebras with one binary operation and one constant.

n	Number of Types
1	1
2	16
3	9,882

6. **Unsolved problems.** While the present paper presents a complete solution to the problem of enumerating finite algebras, our methods do not directly extend to the problems of enumerating the types of finite algebraic systems, such as monoids, groups and rings. The present approach must be modified to take into account the additional axioms which the operations must satisfy. Clearly, associativity should be studied first.

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UNIVERSITY OF CALIFORNIA, BERKELEY