ON DUAL SERIES RELATIONS INVOLVING SERIES OF GENERALIZED BATEMAN K-FUNCTIONS

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1. Here, the solution of the dual series relations

$$\sum_{n=0}^{\infty} \{ A_n/\Gamma(n+l+1)\} K_{2n}^{2l}(x) = f_1(x), \qquad 0 \leqq x < y, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \{ A_n/\Gamma(n+l+(1/2))\} K_{2n}^{2l}(x) = f_2(x), \tag{1.2}$$

$$y < x \leqq \infty, \; l > (-1/2)$$

where $K_{2n}^{2l}(x)$ is then a generalized Bateman K-function, the functions $f_1(x)$, $f_2(x)$ are prescribed, is obtained by reducing the problem of finding the coefficients A_n to that of solving an Abel integral equation.

In recent years, dual series relations involving Fourier-Bessel, Dini series, trigonometric series and series of Jacobi polynomials have been investigated by various workers [1], [2], [5] to [12]. Nere we shall use the method developed by Sneddon and Srivastav for obtaining the solution of dual series relations involving series of generalized Bateman K-functions.

As suggested by Sneddon and Srivastav, with a view to simplify the calculations, we shall consider the problem in two stages. First we assume that $f_2(x) \equiv 0$ and secondly that $f_1(x) \equiv 0$. The solution of the general problem is obtained by adding the two solutions.

2. In this section we list some results for ready reference. Chakrabarty [3] has defined the generalized Bateman K-function as follows.

$$e^{-x/2} K_{2n}^{2l}(x/2) = \frac{(-)^{n-l-1}}{\Gamma(n+l+1)} \left\{\frac{d}{dx}\right\}^{n-l-1} \{e^{-x}x^{n+l}\}, \quad n \geqq l+1, \tag{2.1}$$

$$e^{x} K_{2n}^{2l}(x) = \frac{(-)^{n-l-1}}{\Gamma(2l+2)} (2x)^{2l+1} {}_1F_1[-n+l+1; 2l+2; 2x]. \tag{2.2}$$

A slight modification leads to the equations

$$e^{-x/2} K_{2(n+\alpha)}^{2(l+\alpha)}(x/2)$$

$$= \frac{(-)^{n-l-1}}{\Gamma(n+l+2\alpha+1)} \left\{\frac{d}{dx}\right\}^{n-l-1} \{e^{-x}x^{n+l+2\alpha}\}, \quad n \geqq l+1, \tag{2.3}$$

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$$e^x K_{2(n+\alpha)}^{2(l+\alpha)}(x) = \frac{(-)^{n-l-1}}{\Gamma(2l+2\alpha+2)} (2x)^{2l+2\alpha+1} {}_1F_1[-n+l+1;\, 2l+2\alpha+2;\, 2x]. \tag{2.4}$$

With the help of (2.3) and (2.4) it is easy to prove that

$$\int_0^\infty x^{-(2l+2\alpha+1)} K_{2(n+\alpha)}^{2(l+\alpha)}(x) K_{2(m+\alpha)}^{2(l+\alpha)}(x)\, dx = 2^{2l+2\alpha} \frac{\Gamma(n-l)}{\Gamma(n+l+2\alpha+1)} \delta_{mn}. \tag{2.5}$$

Two particular cases of (2.5) are

$$\int_0^\infty x^{-2l-1} K_{2n}^{2l}(x) K_{2m}^{2l}(x)\, dx = 2^{2l} \frac{\Gamma(n-l)}{\Gamma(n+l+1)} \delta_{mn}, \tag{2.6}$$

$$\int_0^\infty x^{-2l} K_{2n-1}^{2l-1}(x) K_{2m-1}^{2l-1}(x)\, dx = 2^{2l-1} \frac{\Gamma(n-l)}{\Gamma(n+l)} \delta_{mn}, \tag{2.7}$$

where δ_{mn} is the Kronecker delta. The following results are easily derived from the more general results given in [**4**, p. 293 (5), p. 405 (20)]. For $l > (-1/2)$ we have

$$\int_0^x (x-y)^{-1/2} e^y K_{2n}^{2l}(y)\, dy = \{\Gamma(1/2)/2^{1/2}\} e^x K_{2n+(1/2)}^{2l+(1/2)}(x), \tag{2.8}$$

$$\int_0^x (x-y)^{-1/2} e^y K_{2n-1}^{2l-1}(y)\, dy = \{\Gamma(1/2)/2^{1/2}\} e^x K_{2n-(1/2)}^{2l-(1/2)}(x), \tag{2.9}$$

$$\int_x^\infty (y-x)^{-1/2} e^{-y} y^{-l-1} K_{2n}^{2l}(y)\, dy = \{\Gamma(1/2)/2^{1/4}\} \frac{\Gamma(n+l+1/2)}{\Gamma(n+l+1)} x^{-l-(3/4)} e^{-x} K_{2n-(1/2)}^{2l-(1/2)}(x), \tag{2.10}$$

$$\int_x^\infty e^{-y} y^{-l-1} K_{2n}^{2l}(y)\, dy = \{2^{1/2}(n+l)\}^{-1} x^{-l-(1/2)} e^{-x} K_{2n-1}^{2l-1}(x). \tag{2.11}$$

We also require the result

$$\frac{d}{dx}\{e^x K_{2n}^{2l}(x)\} = 2e^x K_{2n-1}^{2l-1}(x) \tag{2.12}$$

of Chakrabarty [**3**].

We note that if $f(x)$ is continuously differentiable then the Abel integral equation

$$f(x) = \int_0^x \frac{\phi(y)}{(x-y)^{1/2}}\,dy \tag{2.13}$$

has a continuous solution

$$\phi(y) = \frac{1}{\pi}\frac{d}{dy}\int_0^y \frac{f(x)}{(y-x)^{1/2}}\,dx. \tag{2.14}$$

Furthermore, if $f(x)$ is continuously differentiable in the interval $[1, \infty)$ then the integral equation

$$f(x) = \int_x^\infty \frac{\phi(y)}{(y-x)^{1/2}}\,dy \tag{2.15}$$

has a continuous solution given by the equation

$$\phi(y) = -\frac{1}{\pi}\frac{d}{dy}\int_y^\infty \frac{f(x)}{(x-y)^{1/2}}\,dx. \tag{2.16}$$

This can easily be established by simple methods (cf. [**13**, p. 229]).

The analysis throughout this paper is purely formal and no attempt is made to justify the interchange of various limiting processes.

3. **Case I.** $f_2(x) \equiv 0$. The dual series equations assume the form

$$\sum_{n=0}^{\infty}\{A_n/\Gamma(n+l+1)\}K_{2n}^{2l}(x) = f_1(x), \qquad 0 \leqq x < y, \tag{3.1}$$

$$\sum_{n=0}^{\infty}\{A_n/\Gamma(n+l+(1/2))\}K_{2n}^{2l}(x) = 0,\ y < x \leqq \infty,\ l > -(1/2). \tag{3.2}$$

Let us suppose that for $0 \leqq x < y$

$$\begin{aligned}\sum_{n=0}^{\infty}&\{A_n/\Gamma(n+l+(1/2))\}K_{2n}^{2l}(x)\\ &= -\frac{e^x x^{2l+1}}{2^{3/4}}\frac{d}{dx}\int_x^y \frac{g_1(u)}{(u-x)^{1/2}}\,du.\end{aligned} \tag{3.3}$$

Using the orthogonal property (2.6), it can be shown that

$$\begin{aligned}A_n = &-\frac{\Gamma(n+l+(1/2)\Gamma(n+l+1)}{2^{2l+(3/4)}\Gamma(n-l)}\\ &\cdot\int_0^y e^x K_{2n}^{2l}(x)\left(\frac{d}{dx}\int_x^y \frac{g_1(u)}{(u-x)^{1/2}}\,du\right)dx.\end{aligned} \tag{3.4}$$

Since

$$(3.5)\quad -\frac{d}{dx}\int_x^y \frac{g_1(u)}{(u-x)^{1/2}}\,du = \frac{g_1(y)}{(y-x)^{1/2}} - \int_x^y \frac{\frac{d}{du}\{g_1(u)\}}{(u-x)^{1/2}}\,du.$$

We obtain with the help of equations (2.8) and (2.12), the equation

$$(3.6)\quad A_n = \frac{\Gamma(n+l+(1/2))\Gamma(n+l+1)\Gamma(1/2)}{\Gamma(n-l)\cdot 2^{2l+(1/4)}}\int_0^u e^u K_{2n-(1/2)}^{2l-(1/2)}(u)g_1(u)\,du,$$

$$n = 0, 1, 2 \cdots.$$

If in equation (3.1), we substitute the coefficients A_n from (3.6) we get, on interchanging the order of summation and integration

$$(3.7)\quad f_1(x) = \int_0^y e^u g_1(u)K_1(u, x)\,du, \qquad 0 \leqq x < y,$$

where

$$(3.8)\quad K_1(u, x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+l+1)\cdot\Gamma(1/2)}{\Gamma(n-l)\cdot 2^{2l+(1/4)}} K_{2n-(1/2)}^{2l-(1/2)}(u)K_{2n}^{2l}(x).$$

With the help of (2.6) and (2.10) it can be shown that

$$(3.9)\quad K_1(u, x) = e^u\cdot u^{l+(3/4)}\cdot e^{-x}\cdot x^l(x-u)^{-1/2}H(x-u),$$

where $H(t)$ is Heaviside's unit function. Thus the equation (3.7) is equivalent to

$$(3.10)\quad e^x x^{-l} f_1(x) = \int_0^x \frac{e^{2u}u^{l+(3/4)}g_1(u)}{(x-u)^{1/2}}\,du, \qquad 0 \leqq x < y.$$

This is an Abel integral equation and its solution is

$$(3.11)\quad e^{2u}u^{l+(3/4)}g_1(u) = \frac{1}{\pi}\frac{d}{du}\int_0^u \frac{e^x x^{-l} f_1(x)}{(u-x)^{1/2}}\,dx.$$

The coefficients A_n may now be computed from relations (3.6) and (3.11).

4. **Case II.** $f_1(x) \equiv 0$. For finding the coefficients satisfying the relations

$$(4.1)\quad \sum_{n=0}^{\infty} \{A_n/\Gamma(n+l+1)\} K_{2n}^{2l}(x) = 0, \qquad 0 \leqq x < y,$$

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+l+(1/2))\} K_{2n}^{2l}(x) = f_2(x), \quad y < x \leqq \infty, \ (l > -(1/2), \tag{4.2}$$

we begin with the assumption that

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+l+1)\} K_{2n}^{2l}(x) = 2^{5/4} e^{-x} x^{l} \int_y^x \frac{g_2(u)}{(x-u)^{1/2}} du, \quad \text{for } y < x \leqq \infty. \tag{4.3}$$

This is equivalent to assuming that

$$A_n = \frac{\Gamma(n+l+1)\Gamma(n+l+(1/2))\Gamma(1/2)}{2^{2l-1}\Gamma(n-l)} \cdot \int_y^{\infty} g_2(u) u^{-l-(3/4)} e^{-u} K_{2n-(1/2)}^{2l-(1/2)}(u)\, du. \tag{4.4}$$

If we multiply both the sides of equation (4.2) by $x^{-l-1}e^{-x}$ and integrate between the limits x to ∞, $y<x\leqq\infty$ we obtain with the help of (2.11) the relation

$$\sum_{n=0}^{\infty} \{A_n/2^{1/2}(n+l)\Gamma(n+l+(1/2))\} x^{-l-(1/2)} e^{-x} K_{2n-1}^{2l-1}(x) = \int_x^{\infty} x^{-l-1} e^{-x} f_2(x)\, dx. \tag{4.5}$$

Substituting the values of the coefficients A_n from (4.4) in equation (4.5) we find on interchanging the order of summation and integration that

$$x^{l+(1/2)} e^{x} F_2(x) = \int_y^{\infty} g_2(u) u^{-l-(3/4)} e^{-u} K_2(u, x)\, du, \quad y < x \leqq \infty, \tag{4.6}$$

where

$$K_2(u, x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+l)\Gamma(1/2)}{2^{2l-1/2}\Gamma(n-l)} K_{2n-(1/2)}^{2l-(1/2)}(u) \cdot K_{2n-1}^{2l-1}(x) \tag{4.7}$$

and

$$F_2(x) = \int_x^{\infty} e^{-x} x^{-l-1} f_2(x)\, dx. \tag{4.8}$$

From (2.7) and (2.9) it easily follows that

$$K_2(u, x) = x^{2l} e^{x-u} (u-x)^{-1/2} H(u-x). \tag{4.9}$$

Consequently the equation (4.6) reduces to the integral equation

$$(4.10)\quad x^{(1/2)-l}F_2(x) = \int_x^\infty \frac{e^{-2u}u^{-l-(3/4)}}{(u-x)^{1/2}}\, g_2(u)\, du, \qquad y < x \leqq \infty.$$

The solution of this integral equation is

$$(4.11)\quad g_2(u) = -\frac{e^{2u}u^{l+(3/4)}}{\pi}\frac{d}{du}\int_u^\infty \frac{x^{(1/2)-l}F_2(x)}{(x-u)^{1/2}}\, dx.$$

The coefficients A_n are given by the relations (4.4) and (4.11).

5. **Added in proof.** The method given above, however, involves sophisticated assumptions and intricate calculation. In this section a much simpler method of solving the problem discussed above is given.

Multiply (1.1) by e^x, differentiate with respects to x, use (2.12) to obtain

$$(5.1)\quad \sum_{n=0}^\infty \{A_n/\Gamma(n+l+1)\} K_{2n-1}^{2l-1}(x)\cdot e^x = \frac{1}{2}\frac{d}{dx}\{e^x f_1(n)\}, \qquad 0 < x \leqq y.$$

By using equations (2.9) and (2.10), we can write the equations (5.1) and (1.2) in the form

$$(5.2)\quad F_1(u) = \frac{e^{-u}}{(2\pi)^{1/2}}\int_0^u (u-x)^{-1/2}\frac{d}{dx}\{e^x f_1(x)\}\, dx = \sum_{n=0}^\infty \{A_n/\Gamma(n+l+1)\} K_{2n-1/2}^{2l-1/2}(u), \qquad 0 \leqq u < y,$$

$$(5.3)\quad F_2(u) = \frac{2^{1/4}}{\Gamma(\frac{1}{2})} u^{l+3/4}e^u \int_u^\infty (x-u)^{-1/2}e^{-x}x^{-l-1}f_2(x)\, dx = \sum_{n=0}^\infty \{A_n/\Gamma(n+l+1)\} K_{2n-1/2}^{2l-1/2}(u), \qquad y < u \leqq \infty.$$

The orthogonal property (2.5) gives the coefficients A_n which are calculated from the equation

$$A_n = \frac{\Gamma(n+l+1)\Gamma(n+l+\frac{1}{2})}{2^{2l-1/2}\Gamma(n-l)} \left[\int_0^y u^{-2l-1/2} K_{2n-1/2}^{2l-1/2}(u) F_1(u)\,du + \int_y^\infty u^{-2l-1/2} K_{2n-1/2}^{2l-1/2}(u) F_2(u)\,du \right]. \tag{5.4}$$

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