ON THE STABILITY OF A PERTURBED NONLINEAR SYSTEM

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Consider the systems of ordinary differential equations

$$(1°)\qquad y' = A(t)y, \qquad \left(' = \frac{d}{dt}\right),$$

$$(2°)\qquad x' = A(t)x + g(t, x),$$

$$(1)\qquad y' = f(t, y),$$

$$(2)\qquad x' = f(t, x) + g(t, x),$$

where each system has a right-hand side continuous on $D = \{(t, x): t \geqq 0,\ x \in E^n\}$ and sufficiently smooth there for the uniqueness of all solutions. For $x \in E^n$, let $|x| = |x_1| + \cdots + |x_n|$. The solution of (1) through some point $(t_0, x_0) \in D$ will be denoted by $y(t, t_0, x_0)$; the solution of (2) by $x(t, t_0, x_0)$. A solution $y(t)$ of (1) exists on some maximal interval $[0, \beta)$ or (α, β), where $0 \leqq \alpha < \beta \leqq +\infty$, and is said to be

bounded if it exists and is bounded on $[0, \infty)$,

bounded in the future if $\beta = +\infty$ *and if for every* $\gamma > \alpha$, $y(t)$ *is bounded on* $[\gamma, \infty)$,

stable if $\beta = +\infty$ *and if for every* $t_0 > \alpha$ *(or* $t_0 \geqq 0$*) and every* $\epsilon > 0$, *there exists* $\delta(\epsilon, t_0) > 0$ *such that for all* $\bar{y}_0 \in E^n$ *satisfying* $|\bar{y}_0 - y(t_0)| < \delta$, *we have* $|y(t, t_0, \bar{y}_0) - y(t)| < \epsilon$, *for all* $t \geqq t_0$,

uniformly stable if it is stable and δ *is independent of* t_0.

Notice that $y(t)$ can be uniformly stable and yet not exist on $[0, \infty)$. If $y(t)$ does exist on $[0, \infty)$, then uniform stability and stability are different concepts, but boundedness and boundedness in the future are equivalent.

Consider the following hypotheses:

H_1°: $A(t)$ is continuous on $[0, \infty)$.

H_1: The Jacobian matrix $f_y(t, y)$ is continuous on D.

H_2°: All solutions of (1°) are uniformly stable.

H_2: All solutions of (1) are uniformly stable, at least one is bounded, and for every $M > 0$, there exists $N(M) > 0$ such that if $|z| \leqq M$, then $|R(t, t_0, z)| \leqq N$ for all $t \geqq t_0 \geqq 0$, where $R(t, t_0, z)$ is the Jacobian matrix $(\partial y_j/(t, t_0, z)\partial z_i)$ of partial derivatives of the solution $y(t, t_0, z)$ of (1).

H_3: For every $M > 0$, there exists a continuous function $h_M(t) \geqq 0$

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such that if $t \geqq 0$ and $|x| \leqq M$, then $|g(t, x)| \leqq h_M(t)$, where $\int_0^\infty h_M(t)dt < \infty$.

Onuchic [1] recently proved

THEOREM 1°. *Let* (1°) *satisfy* $\overset{\circ}{H}_1$ *and* $\overset{\circ}{H}_2$. *Let* $g(t, x)$ *satisfy* H_3. *Then* (a) *for every solution* $y(t)$ *of* (1°), *there exist* $T \geqq 0$ *and* $\delta > 0$ *such that if* $t_0 \geqq T$ *and* $|y(t_0) - x_0| < \delta$, *then the solution* $x(t, t_0, x_0)$ *of* (2°) *is bounded in the future*; *and* (b) *every solution of* (2°) *which is bounded in the future is stable.*

It is the purpose of this note to generalize Onuchic's theorem to perturbed nonlinear systems.

THEOREM 1. *Let* (1) *satisfy* H_1 *and* H_2. *Let* $g(t, x)$ *satisfy* H_3. *Then* (a) *and* (b) *hold for* (1) *and* (2).

Before proving Theorem 1 we make several remarks.

1. If $f(t, y) = A(t)y$, then $R(t, t_0, z) = Y(t)Y^{-1}(t_0)$, where $Y(t)$ is a fundamental matrix for (1°). In this case, $\overset{\circ}{H}_2$ holds if and only if there exists a constant $K > 0$ such that $|Y(t)Y^{-1}(t_0)| \leqq K$ for all $t \geqq t_0 \geqq 0$ (see [2, p. 45]). Furthermore, the zero solution is certainly bounded. Thus $\overset{\circ}{H}_2$ and H_2 are equivalent in the linear case. Clearly, $\overset{\circ}{H}_1$ and H_1 are equivalent, also. Hence Theorem 1° is a special case of Theorem 1.

2. Onuchic used the Tychonoff fixed point theorem to prove Theorem 1°. Thus, for the linear case, our proof gives a new proof of Onuchic's result.

3. In general, H_2 implies that *all* solutions of (1) are bounded in the future. To see this, consider any $t_0 \geqq 0$. Define

$$B = \{y_0 \in E^n \colon y(t, t_0, y_0) \text{ is bounded in the future}\}.$$

The existence of a bounded solution of (1) implies that B is not empty, while the stability of all solutions shows that both B and its complement are open sets. Since E^n is connected, $B = E^n$.

4. If we omit the statement "at least one solution is bounded" from H_2, Theorem 1 may become false, as can be seen by the nonlinear example

$$y' = 1, \tag{3}$$

$$x' = 1 + 0, \tag{4}$$

where $R(t, t_0, z) \equiv 1$, all solutions of (3) are uniformly stable, but none are bounded. Here, (a) is false and (b) is vacuous.

5. An example of a nonlinear system satisfying H_2 is the first order equation

$$y' = -\tfrac{1}{2}y^3 \tag{5}$$

with solution $y(t, t_0, y_0) = y_0[y_0^2(t-t_0)+1]^{-(1/2)}$, so that $R(t, t_0, z) = [z^2(t-t_0)+1]^{-(3/2)}$. Hence $|R(t, t_0, z)| \leqq 1$ for all $t \geqq t_0 \geqq 0$ and all real z. However, it would be nice to characterize those nonlinear systems for which H_2 holds. This remains an open problem.

6. The following example shows that (a) in Theorem 1 is the best possible general result concerning the existence of solutions of (2) which are bounded in the future. Consider the first order equations

$$y' = 0, \tag{6}$$

$$x' = 0 + e^{-2t}x^3, \tag{7}$$

so that $y(t, t_0, y_0) \equiv y_0$ and

$$x(t, t_0, x_0) = x_0[x_0^2(e^{-2t} - e^{-2t_0}) + 1]^{-(1/2)}.$$

Thus $x(t, 0, 1) = e^t$ and $x(t, 0, -1) = -e^t$ which are not bounded in the future. In fact, the solution $x(t, t_0, x_0)$ is bounded in the future if and only if $|x_0| < \exp t_0$. Clearly H_1, H_2, and H_3 hold. Let $y(t)$ be any solution of (6). If $|y(0)| < 1$, then the solution $x(t, t_0, x_0)$ of (7) is bounded in the future if $t_0 \geqq 0$ and $|x_0 - y(t_0)| < 1 - |y(0)|$; that is, we could choose $T = 0$ and $\delta = 1 - |y(0)| > 0$. If $|y(0)| \geqq 1$, then $x(t, t_0, x_0)$ is bounded in the future if $t_0 > \log |y(0)|$ and $|x_0 - y(t_0)| < \exp t_0 - |y(0)|$; that is, we could choose $T = \log |y(0)| + 1 > 0$ and $\delta = |y(0)|(e-1) > 0$. However, if $|y(0)| \geqq 1$, then no such $\delta > 0$ exists for $t_0 \leqq \log |y(0)|$.

PROOF OF THEOREM 1. Fundamental to this proof is the following nonlinear analog of the "variation of constants" formula: for any $(t_0, x_0) \in D$, let $y(t, t_0, x_0)$ and $x(t, t_0, x_0)$ be the corresponding solutions of (1) and (2). Then for as long as they both exist,

$$\begin{aligned} x(t, t_0, x_0) &= y(t, t_0, x_0) \\ &\quad + \int_{t_0}^{t} R(t, s, x(s, t_0, x_0))g(s, x(s, t_0, x_0))ds. \end{aligned} \tag{8}$$

(I believe this is due to Alekseev [**3**] and it can be proved by computing $(d/ds)y(t, s, x(s, t_0, x_0))$ and integrating from t_0 to t.)

Let $y(t)$ be any solution of (1). By Remark 3 above, $y(t)$ is bounded in the future. Let $\bar{\gamma}$ be any point in its maximal interval. Then there is a constant M such that $|y(t)| \leqq M$ on $[\bar{\gamma}, \infty)$. Take $\epsilon = 1$; choose a corresponding δ by the uniform stability of $y(t)$. Choose $N=N(M+2)>0$ by H_2. Choose $T\geqq\bar{\gamma}$ so large that

$$\int_T^\infty h_{M+2}(s)ds < N^{-1}.$$

(Note that T and δ both depend on $y(t)$, in general.) Let $t_0\geqq T$ and $|y(t_0)-x_0|<\delta$. Then $|y(t, t_0, x_0)|\leqq M+1$ on $[t_0, \infty)$. For as long as $|x(t, t_0, x_0)|\leqq M+2$ for $t\geqq t_0$, we have by (8)

$$\begin{aligned} |x(t, t_0, x_0)| &\leqq |y(t, t_0, x_0)| + \int_{t_0}^t |R(t, s, x(s, t_0, x_0))g(s, x(s, t_0, x_0))| ds \\ &\leqq M + 1 + \int_{t_0}^t N h_{M+2}(s)ds \\ &\leqq M + 1 + N\int_T^\infty h_{M+2}(s)ds < M + 2. \end{aligned}$$

Thus $|x(t, t_0, x_0)|<M+2$ on $[t_0, \infty)$, proving (a).

To prove (b), let $x(t)$ be any solution of (2), defined on some maximal interval (α, ∞), which is bounded in the future (the proof when $x(t)$ is defined on $[0, \infty)$ is even easier). Let $\gamma>\alpha$. Then there is a constant M such that $|x(t)|\leqq M$ on $[\gamma, \infty)$. Let $\epsilon>0$. Choose $N=N(M)$ and $N_1=N(M+\epsilon)$ by H_2. Then choose $T>0$ such that

$$\int_T^\infty h_M(s)ds < \frac{\epsilon}{3N} \quad \text{and} \quad \int_T^\infty h_{M+\epsilon}(s) < \frac{\epsilon}{3N_1}\cdot$$

Let $t_0\geqq T$ and $x_0=x(t_0)$. (Incidentally, note that (8) immediately implies

$$|x(t, t_0, x_0) - y(t, t_0, x_0)| \leqq N\int_{t_0}^t h_M(s)ds < \frac{\epsilon}{3}$$

which shows that all solutions of (2) which are bounded in the future are close to certain solutions of (1) uniformly in t for all large enough t. This is a type of asymptotic equivalence result which was also obtained by Onuchic for (1°) and (2°) in [1].) Choose $\delta=\delta(\epsilon/3, y(t, t_0, x_0))>0$ by the uniform stability of $y(t, t_0, x_0)$. Thus δ depends only on ϵ and t_0. Let $|x_0-\bar{x}_0|<\delta$. Then

$$|y(t, t_0, \bar{x}_0) - y(t, t_0, x_0)| < \frac{\epsilon}{3}$$

for all $t \geqq t_0$. Also, for as long as $|x(t, t_0, \bar{x}_0)| < M+\epsilon$ for $t \geqq t_0$,

$$
\begin{aligned}
&| x(t, t_0, \bar{x}_0) - x(t, t_0, x_0) | \\
(9) \qquad &\leqq | y(t, t_0, \bar{x}_0) - y(t, t_0, x_0) | + \int_{t_0}^{t} [N_1 h_{M+\epsilon}(s) + N h_M(s)] ds \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

Thus $|x(t, t_0, \bar{x}_0)| < M+\epsilon$ on $[t_0, \infty)$, hence (9) holds on $[t_0, \infty)$. We must now find δ for $t_0 < T$. Let $\delta_1 = \delta(\epsilon, T)$ as just obtained. If $\gamma \leqq t_0 < T$, choose $\delta = \delta(\delta_1, t_0, T) = \delta(\epsilon, t_0) > 0$ so that if $|\bar{x}_0 - x_0| < \delta$, then

$$| x(t, t_0, \bar{x}_0) - x(t, t_0, x_0) | < \min(\epsilon, \delta_1)$$

for all $t_0 \leqq t \leqq T$ by continuous dependence. Therefore $|x(T, t_0, \bar{x}_0) - x(T, t_0, x_0)| < \delta_1$ so that (9) implies $|x(t, t_0, \bar{x}_0) - x(t, t_0, x_0)| < \epsilon$ on $[T, \infty)$. Thus $|x(t, t_0, \bar{x}_0) - x(t, t_0, x_0)| < \epsilon$ on $[t_0, \infty)$.

We have thus produced $\delta(\epsilon, t_0)$ for this ϵ and any $t_0 \geqq \gamma$. Since ϵ is arbitrary, $x(t)$ is stable on $[\gamma, \infty)$. Since γ is arbitrary, $x(t)$ is stable on (α, ∞), completing the proof.

References

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