ON TOPOLOGY INDUCED BY MEASURE

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Let $(X, \mathfrak{M}, \mu)$ be a complete σ-finite measure space with $\mu(X)>0$. The purpose of this note is to record the existence of a topology on X with the property that each measurable (extended) real-valued function f on X is equal almost everywhere to a unique continuous (extended) real-valued function f^* on X. The mapping $f \to f^*$ provides a "lifting," in the sense that it preserves the algebraic operations (where they are defined). The Stone-Čech compactification of this topology is also discussed.

If $\mathfrak{N}$ denotes the σ-ideal in $\mathfrak{M}$ of the sets of measure zero, then by a result of Maharam ([3]; see also [6]), there exists a mapping ϕ: $\mathfrak{M}/\mathfrak{N} \to \mathfrak{M}$ such that

(1) $\phi(0)=\varnothing$ and $\phi(1)=X$,

(2) $\phi(p \cap q)=\phi(p) \cap \phi(q)$,

(3) $\phi(p \cup q)=\phi(p) \cup \phi(q)$,

(4) $\phi(p) \in p$,

for all $p, q \in \mathfrak{M}/\mathfrak{N}$. By (1) and (2) the sets $\phi(p)$ provide a basis for a topology on X, and this is the topology in question.

THEOREM. *Each continuous function on X is measurable. For each measurable function f on X, there exists a unique continuous function f^* on X which agrees almost everywhere with f. The mapping $f \to f^*$ preserves the algebraic operations* (*where they are defined*).

PROOF. For the first statement, it is sufficient to show that each open set is measurable. The following proof of this is taken from [3]. Any basic open set is measurable since the range of ϕ is in $\mathfrak{M}$. Consider any family $\{\phi(p_\alpha)\}$ of basic open sets. Since the measure space is σ-finite, $\mathfrak{M}/\mathfrak{N}$ satisfies the countable chain condition and is a complete lattice. Hence $p=\bigcup p_\alpha$ exists, and there is a sequence $\alpha_1, \alpha_2, \cdots$ of indices with $p=\bigcup p_{\alpha_i}$. Since ϕ is monotone (by (2) or (3)),

$$\bigcup \phi(p_{\alpha_i}) \subset \bigcup \phi(p_\alpha) \subset \phi(p).$$

But $\phi(p_{\alpha_i}) \in p_{\alpha_i}$ by (4), so $\bigcup\phi(p_{\alpha_i}) \in \bigcup p_{\alpha_i}=p$. Also $\phi(p) \in p$, hence $\phi(p)-\bigcup\phi(p_{\alpha_i})$ is a null set, and therefore $\bigcup\phi(p_\alpha)$ is measurable.

Now let f be a measurable function on X, and define

$$f^*(x) = \sup\{r \mid x \notin \phi[f^{-1}[-\infty, r)]\}$$

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for each $x \in X$. Here we are denoting by $[A]$ the element of $\mathfrak{M}/\mathfrak{N}$ corresponding to $A \in \mathfrak{M}$. It is easy to verify that

$$f^{*-1}[-\infty, r) = \bigcup_{s<r} \phi[f^{-1}[-\infty, s)]$$

and

$$f^{*-1}(r, \infty] = \bigcup_{s<r} (X - \phi[f^{-1}(-\infty, s]).$$

Assumptions (1)–(3) imply that if p' is the complement of p, then $\phi(p')$ is the complement of $\phi(p)$, and therefore f^* is continuous. Since the above unions need only be extended over rational s, the symmetric difference of $f^{*-1}[-\infty, r)$ and $f^{-1}[-\infty, r)$ is a null set for each real r. Consequently f and f^* differ only on a null set. For the uniqueness, it is sufficient to observe that each nonvoid open set contains a basic open set $\phi(p)$ with $p \neq 0$, and therefore has positive measure. That the algebraic operations are preserved is an immediate consequence of the uniqueness.

Our second observation concerns the Stone-Čech compactification of X. In order to discuss this, we introduce the Stone space Y of the complete Boolean algebra $\mathfrak{M}/\mathfrak{N}$. The points of Y are the ultrafilters (maximal dual ideals) $\mathfrak{F}$ of $\mathfrak{M}/\mathfrak{N}$, and the basic open sets in Y are the sets

$$\gamma(p) = \{\mathfrak{F} \mid p \in \mathfrak{F}\}$$

as p varies over $\mathfrak{M}/\mathfrak{N}$. With this topology Y is an extremally disconnected compact Hausdorff space, and the lattice $\mathfrak{M}/\mathfrak{N}$ is isomorphic to the lattice of all open and closed subsets of Y. For each $x \in X$ define

$$F(x) = \{p \mid p \in \mathfrak{M}/\mathfrak{N} \text{ and } x \in \phi(p)\}.$$

A similar construction is used by Donoghue in [1].

LEMMA. *$F(x)$ is an ultrafilter in $\mathfrak{M}/\mathfrak{N}$. The mapping $F: X \to Y$ satisfies $\phi = F^{-1} \circ \gamma$, and consequently the topology in X is the weakest such that F is continuous.*

PROOF. Since $\phi(0) = \emptyset$ by (1), we have $0 \notin F(x)$. On the other hand $\phi(1) = X$, so $1 \in F(x)$ for all x. If $p, q \in F(x)$, then $x \in \phi(p) \cap \phi(q) = \phi(p \cap q)$ by (2), so $p \cap q \in F(x)$. Finally, if $p \in \mathfrak{M}/\mathfrak{N}$ and p' is its complement, then

$$\phi(p) \cup \phi(p') = \phi(p \cup p') = \phi(1) = X$$

by (3) and (1). Thus either $x \in \phi(p)$ or $x \in \phi(p')$, so that either

$p \in F(x)$ or $p' \in F(x)$. Therefore $F(x)$ is an ultrafilter, and $F(x) \in Y$.

Concerning the second assertion of the lemma, for each $p \in \mathfrak{M}/\mathfrak{N}$ and $x \in X$ the following statements are equivalent: $x \in (F^{-1} \circ \gamma)(p)$, $F(x) \in \gamma(p)$, $p \in F(x)$, and $x \in \phi(p)$.

We remark that X is a completely regular space, and that it is Hausdorff when F is one-to-one.

THEOREM. *Let K be a compact Hausdorff space, and let α: $X \to K$ be continuous. Then there exists a unique continuous $\bar{\alpha}$: $Y \to K$ such that $\alpha = \bar{\alpha} \circ F$.*

PROOF. We begin by observing that $F(X)$ is dense in Y. For if the basic open set $\gamma(p)$ is nonempty, then $p \neq 0$, $\phi(p)$ is nonempty, and $F(x) \in \gamma(p)$ for any $x \in \phi(p)$. Moreover, Y is extremally disconnected, so that $F(X)$ is C^*-embedded [2, p. 96]. Therefore Y is the Stone-Čech compactification of $F(X)$.

Next we note the existence of a continuous function β: $F(X) \to K$ such that $\alpha = \beta \circ F$. It follows easily from the lemma that $F(x_1) = F(x_2)$ implies $\alpha(x_1) = \alpha(x_2)$, so that the function β: $F(x) \to \alpha(x)$ is well defined. The lemma also implies that F is open in its range, so that if U is an open subset of K, then $\beta^{-1}(U) = F(\alpha^{-1}(U))$ is open. Thus β is continuous.

Finally, by the first paragraph of the proof, β has a continuous extension $\bar{\alpha}$ to all of Y, and clearly $\alpha = \bar{\alpha} \circ F$. The density of $F(X)$ implies the uniqueness assertion, and completes the proof.

REMARKS. 1. A proof of the above theorem may be based on the following construction. If g is a continuous (extended) real-valued function on X, let

$$\bar{g}(\mathfrak{F}) = \sup\{r \mid \mathfrak{F} \notin \gamma[g^{-1}[-\infty, r)]\}$$

for each $\mathfrak{F} \in Y$. Then $\bar{g}$ is a continuous function on Y satisfying $g = \bar{g} \circ F$. If g is merely measurable, then $g^* = \bar{g} \circ F$ is the continuous function constructed in the first theorem.

2. The mapping $f \to (f^*)^-$ from $L^\infty(X, \mathfrak{M}, \mu)$ into $C(Y)$ is easily seen to be an isometric isomorphism, and is in fact the Gelfand transform.

3. In [7] the topology with basis consisting of the sets $\phi(p) - N$, where $N \in \eta$, is considered. This is stronger than the above topology, but admits no more continuous functions.

4. Those measure spaces for which a mapping ϕ satisfying (1)–(4) exists are characterized in [4]. If the space is not σ-finite, then a continuous function may fail to be measurable, but the remainder of the

first theorem is valid in this context. For the second theorem the completeness of $\mathfrak{M}/\mathfrak{N}$ appears essential.

References

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