Since $g_2=0$, $g_3=1$, it is easy to prove inductively that $C_n=0$ unless $n\equiv 0$ modulo 3. Substitution in (2) shows that $zf(z)$ is a function of z^3, so that

$$f(\eta z) = \eta^2 f(z), \quad \eta^3 = 1.$$

This shows that $f(-\eta^2 h(z)) = \eta f(-h(z))$, and the proof of the equivalence of the various expressions for G in (5) is complete.

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ON THE BOUNDARY BEHAVIOR OF FUNCTIONS MEROMORPHIC IN THE UNIT DISK

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1. **Introduction.** Let $f(z)$ be meromorphic in D: $\{|z|<1\}$, and suppose that the values assumed by $f(z)$ in D lie in a domain G whose boundary Γ has positive logarithmic capacity. Then $f(z)$ is of bounded characteristic in D and has finite radial limits $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ at almost all points $e^{i\theta}$ on C: $\{|z|=1\}$. (For this and more general theory of meromorphic functions, see [**4**, pp. 208 ff.].) The class of functions satisfying these conditions and having the additional property that $f(e^{i\theta})$ belongs to Γ almost everywhere on C has been studied by O. Lehto [**3**] and D. A. Storvick [**6**], who called it *class* (L).

If A is a sequence of points in D satisfying $\sum_{a\in A}(1-|a|)<\infty$, the Blaschke product with respect to A in D is the function $B(z;A) = \prod_{a\in A}[|a|(a-z)/a(1-\bar{a}z)]$. The present note arises from a suggestion by Professor Storvick that the following theorem, established in [**1**], be extended to functions in class (L). Here we denote by A' the derived set of A.

THEOREM 1. *Let E be a set on C. A necessary and sufficient condition*

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that there exist a Blaschke product $B(z;\, A)$ *for which* $B(e^{i\theta}) = \lim_{r\to 1} B(re^{i\theta};\, A)$ *is defined and of modulus one at every point of* C *and such that* $A' = E$ *is that* E *be closed and nowhere dense on* C.

2. Let G be a domain whose boundary Γ has positive logarithmic capacity.

THEOREM 2. *Let* $f(z)$ *be a function of class* (L) *with respect to* G *and* Γ, *and suppose that* $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ *exists and belongs to* Γ *for every* $e^{i\theta}$ *on* C. *Then if* a *is any point of* G *and* $A = \{z \in D : f(z) = a\}$, A' *is closed and nowhere dense on* C.

We note first that A is not empty, since O. Lehto [**3**, p. 12] showed that any omitted value of f in G is a radial limit for $f(z)$. Also from [**3**, p. 12], if A is finite then a must be a boundary value for $f(z)$, unless G is simply-connected and $f(z) = \rho[R(z)]$ for $R(z)$ rational, $|R(z)| \leqq 1$, $|R(e^{i\theta})| = 1$, and $w = \rho(z)$ a schlicht function mapping D onto G. In the latter case $f(z)$ assumes each of its values only finitely many times, and the theorem is clearly true.

For the case when A is infinite, $A' \cap D = \emptyset$ since $f(z)$ is meromorphic in D, and A' is closed. For any point $e^{i\theta}$ of A', $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ is a point $\gamma(\theta)$ of Γ. That is, the radial cluster set, $C_r(f, e^{i\theta})$, for f at $e^{i\theta}$ is a single point $\gamma(\theta)$. However, the interior cluster set, $C_D(f, e^{i\theta})$, for f at $e^{i\theta}$ contains at least the points a and $\gamma(\theta)$. Thus for $e^{i\theta}$ in A' we have $C_D(f, e^{i\theta}) \neq C_r(f, e^{i\theta})$. By a theorem of E. F. Collingwood [**2**, p. 378], A' must be a set of category I on C. Since A' is closed, A' is necessarily nowhere dense on C. (The method of proof that A' is nowhere dense on C was originally suggested to the author by Professor K. Noshiro for use in the proof of Theorem 1.)

3. In the special case that G is simply-connected and its boundary Γ is a Jordan curve, we prove the following

THEOREM 3. *Let* a *be any point of a simply-connected domain* G *whose boundary* Γ *is a Jordan curve. Let* E *be a closed nowhere dense set on* C. *Then there exists a function, analytic in* D, *such that*: (i) $f(z)$ *assumes its values in* G; (ii) *for every* $e^{i\theta}$ *on* C *the limit* $\lim_{r\to 1} f(re^{i\theta}) = f(e^{i\theta})$ *exists and belongs to* Γ; (iii) *if* $A = \{z \in D : f(z) = a\}$, *then* $A' = E$.

If $\rho = g(w)$ is a conformal mapping of G onto $\{|\rho| < 1\}$ with $g(a) = 0$, $g'(a) > 0$, then g can be extended to a homeomorphism of $G \cup \Gamma$ onto $\{|\rho| \leqq 1\}$, and we can consider the mapping $w = g^{-1}(\rho)$ as a homeomorphism of $\{|\rho| \leqq 1\}$ onto $G \cup \Gamma$ which is analytic in $\{|\rho| < 1\}$ and assumes values there in G.

By Theorem 1, since E is closed and nowhere dense on C, there exists a Blaschke product $B(z; H)$ in D with radial limits of modulus one at every point of C such that $H' = E$. (Here H is the set of zeros for $B(z; H)$ in D, and $\sum_{h \in H} (1 - |h|) < \infty$.) We let $\rho = B(z; H)$ and define $w = g^{-1}[B(z; H)]$ for z in D.

The image under $B(z; H)$ of D is the disk $\{|\rho| < 1\}$, since W. Seidel [5] showed that any omitted value in $\{|\rho| < 1\}$ is a radial limit value for $B(z; H)$ at some point of C. It is easily verified that $w = f(z)$ is analytic in D and satisfies (i), (ii), and (iii).

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