THE HOMEOMORPHIC TRANSFORMATION OF c-SETS INTO d-SETS[1]

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Unless otherwise specified, the sets with which we shall deal will be subsets of the open interval $I=(0, 1)$. The term *c-set* will be used to describe a set each of whose points is a bilateral condensation point of the set. A *d-set* will be a set that has metric density 1 at each of its points. (Clearly, every d-set is a c-set, but not conversely.)

In this terminology, a necessary and sufficient condition for a function f to belong to the first Baire class and possess the Darboux (intermediate value) property is that for every real number a, the sets

$$E_a = \{x: f(x) < a\} \quad \text{and} \quad E^a = \{x: f(x) > a\}$$

be F_σ sets [**4**, Theorem 1]. The approximately continuous functions can be characterized in a very similar fashion, namely, by replacing "c-sets" by "d-sets." (This follows from the definition and the fact that such functions belong to the first Baire class [**1**, pp. 165 and 181].)

In the light of these two characterizations, the fact that any Baire 1 Darboux function can be transformed by some appropriate homeomorphic change of variable into an approximately continuous function [**3**] leads to the question: can any F_σ c-set be transformed by some homeomorphism of I onto itself into an F_σ d-set?

The purpose of this paper is to prove that the answer to this question is affirmative and, in so doing, to elucidate something of the structure of F_σ c-sets.

The first lemma is the basis of the structure study.

LEMMA 1. *For any first category F_σ c-set C having* 0 *and* 1 *as limit points, there is a countable collection of sets $Q_1, Q_2, \cdots$ having the following properties*:

(1) $\bigcup_{i=1}^\infty Q_i - \{0, 1\} = C$;

(2) $Q_1 \cup \cdots \cup Q_i$ *is a nowhere-dense perfect set for every positive integer i*;

(3) Q_1 *contains* 0 *and* 1;

(4) *for each positive integer $i>1$, Q_i meets the closure of every interval*

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of $I-(Q_1\cup\cdots\cup Q_{i-1})$ *in a nowhere-dense perfect set that includes both the endpoints of the interval; and*

(5) *for each positive integer* $i>1$, $Q_1\cup\cdots\cup Q_{i-1}$ *and* Q_i *have no other common points than these endpoints.*

PROOF. For any set S, let (in this proof only) $[S]$ denote the set of condensation points of S.

That C is the union of a countable number of nowhere-dense closed sets $F_1, F_2, \cdots$ is already known.

Although 0 cannot be in F_1, the fact that C is a c-set permits the construction of a sequence $\{S_i\}$ of disjoint closed intervals converging toward 0, each S_{i+1} being located entirely to the left of S_i, and such that for each i there is an $n(i)$ for which $[S_i\cap F_{n(i)}]$ is uncountable. A similar construction using T_i and $m(i)$ as the analogs of S_i and $n(i)$ is made for the point 1. Then, the set

$$Q_1=[F_1]\cup\bigcup_{i=1}^{\infty}[S_i\cap F_{n(i)}]\cup\bigcup_{i=1}^{\infty}[T_i\cap F_{m(i)}]\cup\{0,1\}$$

is nowhere-dense and perfect. We denote the (open) components of its complement by $I_{1j}=(a_{1j},b_{1j})$, $j=1,2,\cdots$.

Using $F_2\cap\bar{I}_{1j}$, S_{1jk}, $n(1jk)$, T_{1jk}, and $m(1jk)$ as the analogs of F_1, S_i, $n(i)$, T_i, and $m(i)$, we perform an identical process on each $\bar{I}_{1j}$. (If $a_{1j}\in[F_2\cap\bar{I}_{1j}]$, the intervals S_{1jk} are defined to be empty; and similarly if $b_{1j}\in[F_2\cap\bar{I}_{1j}]$, the intervals T_{1jk} are defined to be empty.) Then, we define the set

$$Q_2=\bigcup_{j=1}^{\infty}\left(([F_2]\cap\bar{I}_{1j})\cup\bigcup_{k=1}^{\infty}[S_{1jk}\cap F_{n(1jk)}]\cup\{a_{1j}\}\right.$$
$$\left.\cup\bigcup_{k=1}^{\infty}[T_{1jk}\cap F_{m(1jk)}]\cup\{b_{1j}\}\right);$$

and we denote the (open) components of $I-(Q_1\cup Q_2)$ by $I_{2j}=(a_{2j},b_{2j})$, $j=1,2,\cdots$.

In this manner, we define for each $i>1$ the set

$$Q_i=\bigcup_{j=1}^{\infty}\left(([F_i]\cap\bar{I}_{1jk})\cup\bigcup_{k=1}^{\infty}[S_{ijk}\cap F_{n(ijk)}]\cup\{a_{ij}\}\right.$$
$$\left.\cup\bigcup_{k=1}^{\infty}[T_{ijk}\cap F_{m(ijk)}]\cup\{b_{ij}\}\right).$$

It is easily seen that Q_i satisfies (2), (4), and (5). The fact that C is a c-set necessitates that each of the points in each of the countable sets

$(F_1\cup \cdots \cup F_i)-(Q_1\cup \cdots \cup Q_i)$ be a condensation point of some F_k, $k>i$. Hence, (1) is true.

The next lemma shows that there is essentially no topological difference among nowhere-dense F_σ c-sets.

LEMMA 2. *If S and T are two nowhere-dense F_σ c-sets having* 0 *and* 1 *as limit points, there is a homeomorphism ϕ of I onto itself such that* $T=\phi(S)$.

PROOF. Let $P_1, P_2, \cdots$ and $Q_1, Q_2, \cdots$ be the collections associated by Lemma 1 with S and T, respectively. $I_{i1}, I_{i2}, \cdots$ will denote the (open) components of the complement of $P_1\cup \cdots \cup P_i$.

Every (closed) interval component of $I-S$ is contained in some I_{1j}. Conversely, every I_{1j} must contain an interval component of $I-S$. (In fact, since S is a c-set, each I_{1j} must contain infinitely many interval components of $I-S$.) For each j, let K_{1j} be such a component of greatest length contained in the (open) interval I_{1j}. Similarly, for each j, let K_{2j} be an interval component of $I-S$ of greatest length contained in I_{2j}. In particular, if I_{2j} contains some K_{1t}, then K_{2j} should be chosen to be K_{1t}. In this manner, an interval component K_{ij} of $I-S$ is associated with each I_{ij} in such fashion that:

(1) $K_{ij}\subset I_{ij}$;

(2) there is in I_{ij} no component of $I-S$ of length exceeding that of K_{ij}; and

(3) if I_{ij} contains some K_{st}, where $s<i$, then K_{ij} is the same as K_{st}.

For any given interval component K of $I-S$, there are only a finite number of components of $I-S$ having length greater than or equal to that of K. The nature of the sets P_i and the fact that the interval components of $I-S$ do not abut require that for i sufficiently large K be in a different component of the set $I-(P_1\cup \cdots \cup P_i)$ than each of the components of $I-S$ having length greater than or equal to that of K. Therefore, the collection of K_{ij} includes all the interval components of $I-S$.

There is a homeomorphism ϕ_1 of I onto itself such that

(1) $\phi_1(P_1)=Q_1$; and

(2) $\phi_1(K_{1j})=L_{1j}$, a (closed) interval component of $I-T$ of greatest length contained in $\phi_1(I_{1j})$, for all j.

Similarly, by working on the closure of each interval of $I-(P_1\cup P_2)$, we can produce a homeomorphism ϕ_2 of I onto itself such that

(1) $\phi_2(x)=\phi_1(x)$ if x is in P_1 or in any K_{1j};

(2) $\phi_2(P_2)=Q_2$; and

(3) $\phi_2(K_{2j})=L_{2j}$, an interval component of $I-T$ of greatest length contained in $\phi_2(I_{2j})$, for all those j for which K_{2j} is not identical with some K_{1t}.

In this manner, a sequence ϕ_i of homeomorphisms is constructed so that for each integer $i>1$,

(1) $\phi(x)=\phi_{i-1}(x)$ if x is in any P_k, $k\leqq i-1$, or in any K_{kj}, $k\leqq i-1$;

(2) $\phi_i(P_i)=Q_i$;

(3) $\phi_i(K_{ij})=L_{ij}$, an interval component of $I-T$ of greatest length contained in $\phi_i(I_{ij})$, for all those j for which K_{ij} is not identical with some K_{st}, $s<i$.

By an argument similar to that given for the collection of K_{ij}, the collection of L_{ij} can be shown to contain all the interval components of $I-T$. Thus, for every i, $|\phi_m(x)-\phi_n(x)|$ is no greater than the length of the longest component of the complement of

$$\bigcup_{k=1}^{i-1}\left(Q_k\cup\bigcup_j L_{kj}\right)$$

($\bigcup_j$ indicating union over all j for which L_{kj} has been defined), for all m, $n\geqq i$ and for all x. Since the length of the longest of these components approaches 0 as i becomes arbitrarily large, ϕ_i converges uniformly to a nondecreasing continuous function ϕ. But, ϕ preserves the linear order existing among all the points in all the sets K_{ij}, and these points form a dense set. Hence, ϕ is strictly increasing and, therefore, a homeomorphism.

Clearly, $T=\phi(S)$.

In the light of this lemma, the proof that any nowhere-dense F_σ c-set can be transformed into a d-set is little more than exhibiting a nowhere-dense F_σ d-set—an easy matter. Let P be a nowhere-dense perfect set of measure 1/2. The set of points of P at which P has metric density 1 is also of measure 1/2 and contains an F_σ set of the same measure. Consequently, this F_σ set is a d-set. Hence, we can state the following lemma.

LEMMA 3. *If C is any nowhere-dense F_σ c-set, there is a homeomorphism of I onto itself that transforms C into a (nowhere-dense F_σ) d-set.*

We shall now show that there is no topological difference among the first category F_σ c-sets that are dense in I.

LEMMA 4. *If S and T are any two first-category F_σ c-sets that are dense in I, there is a homeomorphism ϕ of I onto itself such that $T=\phi(S)$.*

PROOF. The proof is the same as that of Lemma 2 with the following alterations.

(1) Everything concerned with the K_{ij} and the L_{ij} is deleted.

(2) In proving that ϕ is increasing, the dense set of endpoints of the I_{ij} assumes the role of the dense set formed from the K_{ij} in Lemma 2.

Since any F_σ set of measure 1 contained in the set of irrational points is a dense first-category d-set, the following lemma is a simple corollary of Lemma 4.

LEMMA 5. *If C is any first-category F_σ c-set dense in I, there is a homeomorphism of I onto itself that transforms C into a (first-category F_σ) d-set of measure* 1.

The main result will be based upon Lemmas 3 and 5. The following result will also be used directly, but in a very minor role.

LEMMA 6. *If C is an F_σ c-set dense in I, then there is in C a nowhere-dense F_σ set having* 0 *and* 1 *as limit points.*

PROOF. Since C is an uncountable Borel set, it must contain a (nowhere-dense) perfect set [2]. Hence, there is a homeomorphism ϕ of I onto itself such that $\phi(C)$ has positive measure. Let P be any nowhere-dense set having measure greater than that of $I-\phi(C)$. Then, there is an F_σ d-set F contained in $\phi(C)\cap P$. Hence, $\phi^{-1}(F)$ is a nowhere-dense F_σ set contained in C.

Let a and b be, respectively, the left and right extreme limit points of $\phi^{-1}(F)$. If $a>0$, the above type construction is made on $(0, a/2)$. The process is continued (through possibly denumerably many steps) until 0 is the leftmost limit point of the union of all the F_σ sets constructed. The procedure if $b<1$ is completely analogous. The union of *all* the F_σ sets constructed is the desired subset of C.

We are now prepared to prove the main result.

THEOREM. *If C is any F_σ c-set in I, there is a homeomorphism ϕ of I onto itself that transforms C into an F_σ d-set.*

PROOF. If C is either (1) nowhere-dense or (2) of the first category and dense in the interval defined by the extreme limit points of C, then ϕ can be obtained easily from Lemma 3 or Lemma 5, respectively.

In any other case, let $A_1, A_2, \cdots$ be the (closed) intervals of maximum length in which C is dense; and for each positive integer i, let S_i be a nowhere-dense F_σ c-set contained in A_i and having both the endpoints of A_i as limit points (see Lemma 6). (Let the superscript $^\circ$ denote the interior of a set.) Then the set

$$C' = \left(C - \bigcup_{i=1}^{\infty} A_i^{\circ}\right) \cup \bigcup_{i=1}^{\infty} S_i$$

is easily seen to be a nowhere-dense F_σ set contained in C. By Lemma 2, there is a homeomorphism ϕ_1 such that $\phi_1(C')$ is a (nowhere-dense) F_σ d-set.

Let T_i denote $\phi_1(C \cap A_i^{\circ})$ if this set is of the first category. If this set is of the second category, let T_i be the set formed from it by replacing the interior of each interval component by any first category F_σ c-set dense in that component. Thus, T_i is, in every case, a first category F_σ c-set dense in $\phi_1(A_i)$. On each of the intervals $\phi_1(A_i)$, define the function ϕ_2 to be the analog of the homeomorphism of Lemma 5; and let $\phi_2(x) = x$ elsewhere. Finally, let ϕ be the homeomorphism $\phi_2 \circ \phi_1$. Since the measure of $\phi(T_i)$ is equal to the measure of $\phi(A_i)$ for every i, $\phi(C)$ is an F_σ d-set.

The extension of this theorem to the entire real line should be rather clear. However, the extension to any other Euclidean space, even to the plane, is a very much open problem; and the first step toward its solution must be the formulation of a suitable definition of c-set in the space.

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