ON THE COINCIDENCE OF THREE INVARIANT
σ-ALGEBRAS ASSOCIATED WITH AN AFFINE
TRANSFORMATION

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Introduction. This paper is concerned with an ergodic affine transformation T of a compact metric connected abelian group X. T is therefore a Haar measure preserving homeomorphism of X onto itself and we can associate with T, three invariant σ-algebras or partitions π, α, η: these are the maximum σ-algebras such that the factors T_π, T_α, T_η have zero entropy, quasi-discrete spectrum, the distal property, respectively. It will be shown that the three σ-algebras are identical.

It is assumed that the reader is familiar with the theory of entropy of ergodic measure preserving transformations. For an account of this theory and its notations and for the related notions of measurable partition and factor transformation cf. [1].

Let $(X, \mathcal{E}, m)$ be a measure space point isomorphic to the unit interval with Lebesgue sets and Lebesgue measure. A compact metric abelian group with normalized complete Haar measure is of this type. Let T be an invertible ergodic measure preserving transformation of X onto itself. Associated with each sub-σ-algebra $\mathcal{D} \subset \mathcal{E}$ there is an unique (mod 0) measurable partition δ of X and conversely. ($\mathcal{D} = \mathcal{E}(\delta)$, $\delta = \epsilon(\mathcal{D})$.) $\mathcal{D}$ is finite if and only if δ is finite. The relations $T\mathcal{D} \subset \mathcal{D}$, $T\mathcal{D} = \mathcal{D}$ correspond to the relations $T\delta \leq \delta$, $T\delta = \delta$ (mod 0), and if ν, $\mathfrak{N}$, ϵ are the trivial partition, trivial σ-algebra and partition into one point sets, respectively, then $\nu = \epsilon(\mathfrak{N})$, $\mathfrak{N} = \mathcal{E}(\nu)$, $\epsilon = \epsilon(\mathcal{E})$, $\mathcal{E} = \mathcal{E}(\epsilon)$.

1. *Corresponding to T there is a maximum σ-algebra $\mathcal{E}(\pi) \subset \mathcal{E}$ such that the factor transformation T_π has zero entropy. Moreover $T\mathcal{E}(\pi) = \mathcal{E}(\pi)$. T has zero entropy if and only if $\mathcal{E}(\pi) = \mathcal{E}$ and has completely positive entropy if and only if $\mathcal{E}(\pi) = \mathfrak{N}$. (Pinsker [2]).*

2. *If $\mathcal{E}(\gamma)$, $\mathcal{E}(\delta)$ are doubly invariant σ-algebras ($T\mathcal{E}(\gamma) = \mathcal{E}(\gamma)$, $T\mathcal{E}(\delta) = \mathcal{E}(\delta)$,) and if T_γ, T_δ have zero entropy and completely positive entropy, respectively, then $\mathcal{E}(\gamma)$, $\mathcal{E}(\delta)$ are independent i.e. if $A \in \mathcal{E}(\gamma)$, $B \in \mathcal{E}(\delta)$ then*

$$m(A \cap B) = mA \cdot mB \qquad [2].$$

A homeomorphism T of a compact metric space X is said to be distal if $\lim_{m \to \infty} T^{n_m} x = \lim_{m \to \infty} T^{n_m} y$ implies $x = y$.

3. *There is a minimum closed invariant equivalence relation on X or, alternatively, there is a maximum closed Hausdorf partition η of X such that the factor transformation T_η is distal (Ellis [3]).*

4. *Every homeomorphism which is a continuous image of a distal homeomorphism is distal* [3].

An affine transformation T of a compact abelian group X is a transformation of the form

$$Tx = a + A(x),$$

where A is an automorphism of X and $a \in X$.

For the definition of a measure preserving transformation with quasi-discrete spectrum cf. Abramov [4], and for the definition of a homeomorphism with quasi-discrete spectrum cf. [5].

An ergodic affine transformation T of a compact metric connected abelian group X possesses a maximum partition α such that T_α has quasi-discrete spectrum. The partition α is the same for both interpretations of "quasi-discrete spectrum," and $\mathcal{E}(\alpha)$ is the smallest σ-algebra for which the quasi-eigenfunctions are measurable.

5. *α is the partition of X into cosets of* ann Γ_1, *where* ann Γ_1 *is the annihilator of Γ_1 and Γ_1 is the subgroup of the character group Γ given by*

$$\Gamma_1 = \{\gamma \in \Gamma : (A - I)^n \gamma = 0 \text{ for some } n \geqq 1\} \qquad [6].$$

We shall need the following result:

6. *If A is an automorphism of a compact abelian group (regarded as a Haar measure preserving transformation) such that all orbits (other than that of 0) are infinite, then A has completely positive entropy. Under the same hypothesis, if $Tx = a + A(x)$, then T has completely positive entropy, for it is clear that $(A - I)X = X$ and if $(A - I)\beta = -a$ then*

$$T(\beta + x) = \beta + A(x),$$

i.e. T and A are isomorphic [7].

In the next three statements we assume $Tx = a + A(x)$ is an affine transformation of a compact metric connected abelian group X, regarded as a Haar measure preserving transformation.

7. *T is ergodic if and only if the smallest closed group containing a and $(A - I)X$ is X and the only finite orbits in the character group Γ under A are fixed elements* [8], [9].

8. *If T is ergodic and has quasi-discrete spectrum then T has zero entropy* [4].

9. *If T is ergodic and has quasi-discrete spectrum then T is distal* [5].

Throughout $Tx = a + A(x)$ will be an ergodic affine transformation of a compact metric connected abelian group X, and Γ will denote the character group of X. Automorphisms and their duals will be denoted by the same symbol. $Z(\lambda)$ will denote the ring of polynomials with integer coefficients.

Our results will depend on Propositions 1–9 and we refer to these propositions by ( ), whereas [ ] will denote a paper listed in the reference.

1. Zero-entropy and quasi-discrete spectrum.

THEOREM 1. *The maximum partition π such that T_π has zero entropy and the maximum partition α such that T_α has quasi-discrete spectrum, concide.*

PROOF. In view of (5) and (8) it suffices to show that the partition of X into cosets of ann Γ_1, where $\Gamma_1 = \{\gamma \in \Gamma \colon (A - I)^n \gamma = 0$ for some $n \geq 1\}$ is finer than π. Alternatively we have to show that the Fourier expansion of $\chi_E(x)$ does not depend on characters outside Γ_1, if $E \in \mathcal{E}(\pi)$.

Suppose $\gamma \notin \Gamma_1$ and suppose $p(A)\gamma \neq 0$ for all $0 \neq p \in Z(\lambda)$ then $[A, \gamma]$, the smallest A invariant subgroup of Γ containing γ, has no finite orbits other than the trivial orbit. $[A, \gamma]$ is the character group of $X' = X/\text{ann}[A, \gamma]$ and if A' is the induced automorphism on X' and a' is the coset of ann $[A, \gamma]$ to which a belongs then $T'(x') = a' + A'(x')$ has completely positive entropy (6). Consequently the coset partition of ann $[A, \gamma]$ is independent of π and the Fourier expansion of $\chi_E(x)$ does not depend on γ, if $E \in \mathcal{E}(\pi)$. Therefore the partition into cosets of ann(Γ'') where $\Gamma'' = \{\gamma \in \Gamma \colon p(A)\gamma = 0$ for some $0 \neq p \in Z(\lambda)\}$, is finer than π and we may therefore suppose $\Gamma = \Gamma''$ i.e. we may suppose each $\gamma \in \Gamma$ is annihilated by some nontrivial $p(A)$ where $p \in Z(\lambda)$.

Let us embed Γ into the smallest divisible group Γ' containing Γ. Γ is countable and torsion free and therefore Γ' is countable and torsion free. Moreover for each $\gamma' \in \Gamma'$ there exists an integer $k \neq 0$ such that $k\gamma' \in \Gamma$. The automorphism A can be lifted to an automorphism A' of Γ' in such a way that if $\gamma' \in \Gamma'$ and k is integer such that $k\gamma' \in \Gamma$ then $kA'(\gamma') = A(k\gamma')$.

Thus $Tx = a + A(x)$ can be viewed as a factor of the affine transformation $T'x' = a' + A'(x)$, which acts on X', the character group of Γ', where a' is an element of the coset a of ann(Γ).

Since Γ has only fixed elements as finite A orbits, the same is true of Γ' and A' orbits. To show that T' is ergodic we will show that the

smallest closed group containing a' and $(A'-I)X'$ is X' (7). If this were not so there would exist a nontrivial character γ' such that $(A'-I)\gamma' = 0$ and $\gamma'(a') = 0$ i.e. $\gamma'(a') = 0$, $\gamma'(A'x') = \gamma'(x')$. But since there exists $k \neq 0$ such that $0 \neq \gamma = k\gamma' \in \Gamma$ we would have $\gamma(a') = 0$, $\gamma(A'x') = \gamma(x')$, $\gamma(Ax) = \gamma(x)$. Moreover γ is constant on the cosets of ann Γ and therefore $\gamma(a) = 0$. This contradicts the fact that the smallest closed subgroup of X containing a and $(A-I)X$ is X, (7). T is therefore a factor of the ergodic affine transformation T'.

The character group Γ' of X' is divisible and for every $\gamma' \in \Gamma'$ there exists $0 \neq p \in Z(\lambda)$ such that $p(A')\gamma = 0$. Write $p(\lambda) = (\lambda-1)^n p'(\lambda)$ where $p' \in Z(\lambda)$ and $p'(\lambda)$, $\lambda - 1$ are coprime. Let $u, v \in Z(\lambda)$ be such that

$$u(\lambda)(\lambda - 1)^n + v(\lambda)p'(\lambda) = k$$

for some integer $k \neq 0$, then

$$u(A')(A' - I)^n(\gamma'/k) + v(A')p'(A)(\gamma'/k) = \gamma'.$$

Therefore $\gamma' = \gamma_1' + \gamma_2'$ where $\gamma_1' = v(A')p'(A)(\gamma'/k)$, $\gamma_2' = u(A')(A'-I)^n(\gamma'/k)$ and $p'(A')\gamma_2' = 0$, $(A'-I)^n\gamma_1' = 0$.

Let $\Gamma_2' = \{\gamma' \in \Gamma' : p'(A')\gamma' = 0$ for some $p'(\lambda)$ prime to $\lambda - 1\}$ and let $\Gamma_1' = \{\gamma' \in \Gamma' : (A'-I)^n\gamma' = 0$ for some $n \geq 1\}$. Γ_1', Γ_2' are A' invariant subgroups of Γ' and $\Gamma' = \Gamma_1' \times \Gamma_2'$.

Consequently $X' = X_1' \times X_2'$ where $X_1' = X'/\text{ann } \Gamma_1'$ $X_2' = X'/\text{ann } \Gamma_2'$ and the character groups of X_1', X_2' are Γ_1', Γ_2', respectively. Moreover $T' = T_1' \times T_2'$ where $T_1'(x_1') = a_1' + A_1'(x_1')$, $T_2'(x_2') = a_2' + A_2'(x_2')$ and A_1', A_2' are the automorphisms induced on X_1', X_2' by A', respectively, and a_1', a_2' are the cosets of $\text{ann}(\Gamma_1')$, $\text{ann}(\Gamma_2')$ of X', respectively, to which a' belongs.

From (5), (8) it follows that T_1' has quasi-discrete spectrum and zero entropy. From (6) it follows that T_2' has completely positive entropy. These two statements together with (2) imply that π' (the maximum partition of X' with zero entropy for T') is the partition of X' into cosets of ann Γ_1'.

But $\pi \leq \pi'$ and if $(E, X - E) \leq \pi$ the Fourier expansion of $\chi_E(x')$ depends only on the characters of $\Gamma \cap \Gamma_1'$. Consequently the Fourier expansion of $\chi_E(x)$ depends only on the characters of

$$\Gamma_1 = \{\gamma \in \Gamma : (A - I)^n\gamma = 0 \text{ for some } n \geq 1\},$$

and the proof is complete.

2. The distal property and quasi-discrete spectrum.

THEOREM 2. *The maximum partition η, such that T_η is distal and the maximum partition α such that T_α has quasi-discrete spectrum, coincide.*

PROOF. In view of (9) it suffices to show that T_η has quasi-discrete spectrum. We first note that η is translation invariant by showing that $T_{\eta+x}$ is also distal for each $x \in X$. Let d be a translation invariant metric on X. Since the distal relation is closed and since X is compact metric, it follows that the factor space X_η is also compact metric, [10, p. 148]. Let d' be a metric for X_η, then

$$\lim_{m \to \infty} d'(T_\eta^{n_m} E, T_\eta^{n_m} F) = 0$$

if and only if there exist $x_m \in E$, $y_m \in F$ such that

$$\lim_{m \to \infty} d(T^{n_m} x_m, T^{n_m} y_m) = 0.$$

Consequently, if $x_m \in E \in \eta$, $y_m \in F \in \eta$ then

$$\lim_{m \to \infty} d(T^{n_m} x_m, T^{n_m} y_m) = \lim_{m \to \infty} d(A^{n_m} x + T^{n_m} x_m, A^{n_m} x + T^{n_m} y_m)$$

$$= \lim_{m \to \infty} d(T^{n_m}(x_m + x), T^{n_m}(y_m + x))$$

$$= 0$$

implies $E = F$. Therefore $\eta + x \leq \eta$ for all $x \in X$, and η is a partition of X into cosets of some closed subgroup X' of X. We may therefore suppose that T is a distal ergodic affine transformation and prove that T has quasi-discrete spectrum.

In view of (5) we have to show that for each $\gamma \in \Gamma$, there exists $n \geq 1$ such that $(A - I)^n \gamma = 0$. If this were not so, then either there would exist $\gamma \in \Gamma$ and $p \in Z(\lambda)$ prime to $\lambda - 1$ such that $p(A)\gamma = 0$ or there would exist $\gamma \in \Gamma$ such that for all $0 \neq p \in Z(\lambda)$, $p(A)\gamma \neq 0$. In either case (using (7) and the ergodicity of T) A would have no finite orbits in $[A, \gamma]$ other than that of 0. $[A, \gamma]$ is the character group of $X' = X/\operatorname{ann}[A, \gamma]$ and if $T'x' = a' + A'(x')$ where A' is the automorphism induced on X' by A and a' is the coset of $\operatorname{ann}[A, \gamma]$ to which a belongs, then T' is a continuous image of T. Consequently T' is distal, and since T' is conjugate to A' cf. (6), A' is distal. Since A' is ergodic, for almost all $x' \in X'$ there exist a sequence $\{n_m\}$ such that

$$\lim_{m \to \infty} A'^{n_m}(x') = 0,$$

and obviously $\lim_{m \to \infty} A'^{n_m}(0) = 0$.

Therefore A' is not distal. This contradiction establishes the theorem.

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