ALGEBRAIC GROUP AUTOMORPHISMS HAVING FINITE FIXED POINT SETS

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1. Introduction. The main conclusions of this paper are given in Theorems 7, 10, 11 and 12. These theorems describe some of the structure of a connected linear algebraic group G having a birational automorphism σ such that the subgroup $F_G(\sigma)$ of fixed points of σ in G is finite. (Some of this structure is described in [3] in the case where $F_G(\sigma) = \{1\}$.)

Algebraic group automorphisms having only finitely many fixed points occur significantly more frequently than do those which keep fixed only the identity, and they occur under less restrictive conditions. (One simple indication of this is that over the field of complex numbers, there are tori of positive dimension which admit birational automorphisms of prime period having only finitely fixed points, but every such automorphism has nonidentity fixed points.)

The proof of Theorem 7 is heavily dependent upon Lemmas 2 and 6. Lemma 6 is also important for the proofs of Theorems 10, 11, 12.

2. Preliminaries. In this paper, G is a connected linear algebraic group over an algebraically closed field of characteristic 0 or $p > 0$ and σ is a birational automorphism of G. If G is a normal subgroup of a group K, then for x in K, $\mathrm{Int}_G\, x$ is the automorphism of G defined by: $\mathrm{Int}_G\, x(g) = x^{-1}gx$ for g in G.

LEMMA 1. *Let G be semisimple and let σ be a birational automorphism of G. Then there is a linear algebraic group K containing G as a closed normal subgroup of finite index and an element s in K such that $\sigma = \mathrm{Int}_G\, s$.*

PROOF. Aut $G/\mathrm{Int}_G\, G$ is finite [2, §17-07]. It follows that for some integer n, σ^n lies in $\mathrm{Int}_G\, A$ where A is a closed connected divisible abelian subgroup of G having the property that σ commutes with the elements of $\mathrm{Int}_G\, A$. (For if $\sigma^k = \mathrm{Int}_G\, s$ and if B denotes the closed subgroup of G generated by s, then B is abelian and, as is easily seen, σ commutes with each element of $\mathrm{Int}_G\, B$; moreover for some j, σ^{kj}

lies in $\mathrm{Int}_G\ B_0$, where B_0 is the connected component of the identity in B and has the form $B_0 = TU$ (direct), T being a torus and U a closed connected unipotent subgroup of G; therefore we can let $A = B_0$ if the characteristic is 0, $A = T$ if the characteristic is $p > 0$.) Consequently, there exists τ in Aut G such that $\tau^n = 1$, $\sigma\tau = \tau\sigma$, and $\sigma \in \tau\ \mathrm{Int}_G\ G$. (For if $\sigma^n = \mathrm{Int}\ x^n$ with x in A, we can let $\tau = \sigma\ \mathrm{Int}_G\ x^{-1}$.) Let $K = \langle\tau\rangle G$ (semidirect). Then K can be given the structure of a linear algebraic group containing G as a closed subgroup, since $(K:G)$ is finite (e.g. by way of the representation of K induced by G). Choosing y in G such that $\sigma = \tau\ \mathrm{Int}_G\ y$, we have $\sigma = \mathrm{Int}_G\ s$ where $s = \tau y$ in K.

We need the following lemma (see also Theorem 2 of [6]):

LEMMA 2. *Let G be semisimple. Let τ be a birational automorphism of G which keeps stable a maximal torus T of G and a Borel subgroup B of G containing T. Then* dim $F_G(\tau) \geqq 1$ *if* dim $G \geqq 1$.

PROOF. Suppose that dim $G \geqq 1$. The cyclic group $\langle\tau\rangle$ generated by τ acts in the group T^* of rational characters of T and keeps stable the subset S of fundamental roots of T with respect to B. Let m be the index in T^* of the subgroup generated by S. Let t be an element of finite order such that $\alpha(t) = \beta(t)$ whenever α and β are in the same orbit of $\langle\tau\rangle$ in S. Then $\alpha(\tau(t)) = \alpha(t)$ for α in S. Thus $\chi^m(\tau(t)) = \chi^m(t)$ for χ in T^*. Thus $\chi(\tau(t^m)) = \chi(t^m)$ for χ in T^* and $\tau(t^m) = t^m$ since T^* separates points. The order of t (and hence of t^m) can be taken to be arbitrarily large (since S is a linearly independent subset of T^*). Thus dim $F_T(\tau) \geqq 1$.

LEMMA 3. *Let $\mathfrak{G}$ be a p-group having a normal subgroup $\mathfrak{H}$ which is infinite and has a faithful finite dimensional representation ρ over a field of characteristic p. Then the centralizer in $\mathfrak{H}$ of each element g of $\mathfrak{G}$ is infinite.*

PROOF. Let g be an element of $\mathfrak{G}$. Let $\mathfrak{M} = \langle g\rangle\mathfrak{H}$. Then $(\mathfrak{M}:\mathfrak{H})$ is finite and ρ induces a faithful finite dimensional representation of $\mathfrak{M}$. Thus we may assume without loss of generality that $\mathfrak{M}$ is a linear group. This group consists of unipotent transformations since $\mathfrak{M}$ is a p-group. Thus $\mathfrak{M}$ is nilpotent by Kolchin's theorem [5, pp. 775 and 776]. Let $\mathfrak{H}_0$ be the (Zariski-) connected component of the identity in $\mathfrak{H}$. Then $\mathfrak{H}_0$ is a normal subgroup of $\mathfrak{M}$ of finite index, whence $\mathfrak{H}_0$ is infinite. For $S \subseteq \mathfrak{M}$, let $g_0(S) = S$, $g_1(S) = (g, S)$, $\cdots$, $g_{i+1}(S) = (g, g_i(S))$ $((g, h) = g^{-1}h^{-1}gh)$. Since $\mathfrak{M}$ is nilpotent we can find i such that $g_{i+1}(\mathfrak{H}_0) = \{1\}$ and $g_i(\mathfrak{H}_0) \neq \{1\}$. Since $\mathfrak{H}_0$ is con-

nected, $g_i(\mathfrak{H}_0)$ is connected. Thus $g_i(\mathfrak{H}_0)$ is an infinite subset of $\mathfrak{H}$ centralizing g. Thus the centralizer in $\mathfrak{H}$ of an arbitrary element g of $\mathfrak{G}$ is infinite.

3. Solvability of G when $F_G(\sigma)$ is finite. For x, g in G, let $x_\sigma g = \sigma(g^{-1})xg$. G acts on itself on the right as a group of birational transformations, an element g in G sending x into $x_\sigma g$. Since $F_G(\sigma) = \{g \mid 1_\sigma g = 1\}$, an application of a dimension theorem [1, Theorem 2, p. 106] yields:

LEMMA 4. *$F_G(\sigma)$ is finite if and only if $1_\sigma G = \{1_\sigma g \mid g \in G\}$ is dense in G.*

An immediate consequence of this is:

LEMMA 5. *Let R be a closed connected characteristic subgroup of G. Define $\overline{G} = G/R$ and let $\bar{\sigma}$ denote the automorphism of $\overline{G} = G/R$ induced by σ. Then if $F_G(\sigma)$ is finite, $F_{\overline{G}}(\bar{\sigma})$ is finite.*

If σ has finite order, the semidirect product $\langle \sigma \rangle G$ of the cyclic group $\langle \sigma \rangle$ generated by σ with G may be regarded as a linear algebraic group containing G as a closed normal subgroup of finite index (by way of the representation of $\langle \sigma \rangle G$ induced by G). Elements τ of $\langle \sigma \rangle G$ are regarded as automorphisms of G as follows: $\tau(g) = \tau^{-1}g\tau$ for τ in $\langle \sigma \rangle G$ and g in G.

LEMMA 6. *Let σ have finite order n. Suppose that $F_G(\sigma)$ is finite. Define $K = \langle \sigma \rangle G$ (semidirect). Then for τ in the subset σG of K, the exponent of $F_G(\tau)$ divides n.*

PROOF. $1_\sigma G = \{\sigma^{-1}g^{-1}\sigma g \mid g \in G\}$ is dense in G by Lemma 4. Thus $\sigma(1_\sigma G) = \{g^{-1}\sigma g \mid g \in G\}$ is dense in σG. If $N = \{\tau \in \sigma G \mid \tau^n = 1\}$, then N is closed and $\{g^{-1}\sigma g \mid g \in G\} \subseteq N$. Thus $N = \sigma G$ and $\tau^n = 1$ for τ in σG. Thus for τ in σG and g in $F_G(\tau)$, $1 = (\tau g)^n = \tau^n g^n = g^n$. Thus the exponent of $F_G(\tau)$ divides n.

THEOREM 7. *Let $F_G(\sigma)$ be finite. Then G is solvable.*

PROOF. Suppose first that G is semisimple. Then by Lemma 1, there is a linear algebraic group K containing G as a closed normal subgroup of finite index and an element s in K such that $\sigma = \mathrm{Int}_G s$. If $(K:G) = m$, then $F_G(\sigma)$ contains s^m and s^m has finite order. Thus σ has finite order. Thus we may take K to be the semidirect product $\langle \sigma \rangle G$ described in Lemma 6. Let T be a maximal torus of G, B a Borel subgroup of G containing T. Since $\mathrm{Int}_G G$ acts transitively on the set of Borel subgroups of G, and since $\mathrm{Int}_B B$ acts transitively on the set of maximal tori of B, there exists τ in σG such that B and T are

τ-stable. If dim $G \geqq 1$, then dim $F_T(\tau) \geqq 1$ by Lemma 2. But the exponent of $F_G(\tau)$ is finite by Lemma 6. Thus dim $G = 0$ and $G = \{1\}$.

We now consider the general case and let R be the radical of G. Define $\overline{G}$, $\bar{\sigma}$ as in Lemma 5. Then $F_{\overline{G}}(\bar{\sigma})$ is finite since $F_G(\sigma)$ is finite (Lemma 5). Thus $\overline{G} = \{1\}$ by the preceding discussion. Thus G is solvable.

4. General structure of (G, σ) when $F_G(\sigma)$ is finite.

PROPOSITION 8. *Let $F_G(\sigma)$ be finite. Then $1_\sigma G = G$ and σ keeps stable a maximal torus of G.*

PROOF. G is solvable by Theorem 7. If G is abelian, then the mapping sending g into $1_\sigma g$ is a rational homomorphism from G into G; and therefore $1_\sigma G$ is closed. But $1_\sigma G$ is dense in G by Lemma 4. Thus $1_\sigma G = G$ if G is abelian.

The proof continues by induction on dim G. Let R be a closed connected characteristic abelian subgroup of G of positive dimension (e.g. the next to last term of the derived series of G). Define $\overline{G}$, $\bar{\sigma}$ as in Lemma 5. Then $F_{\overline{G}}(\bar{\sigma})$ is finite by Lemma 5 and $\overline{G} = 1_{\bar{\sigma}}\overline{G}$ by induction. Thus for g in G, there exists x in G such that $\bar{g} = 1_{\bar{\sigma}}\bar{x}$. Then $\bar{1} = \bar{g}_{\bar{\sigma}}\bar{x}^{-1} = [g_\sigma x^{-1}]^-$ and $g_\sigma x^{-1}$ is in R. Since R is abelian, $R = 1_\sigma R$. Thus $g_\sigma x^{-1} = 1_\sigma y$ for some y in R. Then $g = (g_\sigma x^{-1})_\sigma x = (1_\sigma y)_\sigma x = 1_\sigma(yx)$ and g is in $1_\sigma G$. Thus $G = 1_\sigma G$.

Now a standard argument shows that σ keeps stable a maximal torus of G. Let T be a maximal torus of G. Then $\sigma(T) = x^{-1}Tx$ for some x in G. Since $G = 1_\sigma G$, $x = g\sigma(g^{-1})$ for some g in G. Now $\sigma(g^{-1}Tg) = \sigma(g)^{-1}\sigma(T)\sigma(g) = \sigma(g)^{-1}(x^{-1}Tx)\sigma(g) = g^{-1}Tg$ (since $x\sigma(g) = g$), and $g^{-1}Tg$ is σ-stable.

LEMMA 9. *Let $F_G(\sigma)$ be finite and let σ have finite order. Let σ_s be the p-regular part of σ. Then $F_G(\sigma_s)$ consists of semisimple elements.*

PROOF. G is solvable by Theorem 7. Thus $G = TU$ where T is a maximal torus of G and U is a closed connected characteristic unipotent subgroup of G. Let $\sigma = \tau\mu$ where τ, μ are the p-regular and p-singular components of σ respectively. Then $V = F_U(\tau)$ is connected (by [2, §6-02, Corollary 2], since σ is of finite order and $\langle\sigma\rangle G$ can be given the structure of an algebraic linear group containing G as a closed subgroup) and μ-stable. If dim $V \geqq 1$, $F_V(\mu)$ is infinite (apply Lemma 3 to $\mathfrak{G} = \langle\mu\rangle V$ if the characteristic is $p > 0$; if the characteristic is 0, $\mu = 1$). But $F_G(\sigma)$ is finite and $F_G(\sigma)$ contains $F_V(\mu)$. Thus dim $V = 0$ and $F_U(\tau) = \{1\}$. Since U is the set of unipotent elements of G [2, §6-06] and since $F_G(\tau)$ is algebraic, $F_G(\tau)$ consists of semisimple elements.

THEOREM 10. *Let $F_G(\sigma)$ be finite and let σ have finite order. Then σ keeps stable a unique maximal torus T_σ; and T_σ contains $F_G(\sigma)$. Moreover T_σ is the only maximal torus of G stable under the p-regular part σ_s of σ; and T_σ contains $F_G(\sigma_s)$.*

PROOF. By Proposition 8, σ keeps stable some maximal torus T of G. Let g be in $F_G(\tau)$ where $\tau = \sigma_s$. Then since $G = TU$ where U is a closed connected characteristic unipotent subgroup, $g = tu$ with t in T and u in U. But $g = g^\tau = t^\tau u^\tau$ and t^τ is in T, u^τ in U. By the uniqueness of the decomposition $g = tu$, $t = t^\tau$ and $u = u^\tau$. Thus u is in $F_G(\tau)$. By Lemma 9, $u = 1$. Thus $g = t \in T$ and $F_G(\tau) \subseteq T$. Since $\tau \in \langle \sigma \rangle$, $F_G(\sigma) \subseteq F_G(\tau) \subseteq T$.

If S is a second maximal torus stable under τ, then $S = T^x$ for some x in U. We have $T^x = \tau(T^x) = T^{\tau(x)}$ and $\tau(x)x^{-1}$ is in $N = \{u \in U \mid T^u = T\}$ (here we use the notation $T^a = a^{-1}Ta$). Since G is solvable, N is a closed connected group centralizing T see [2, §6-04]. Since T and U are τ-stable and N is the centralizer in U of T, N is τ-stable. Thus $N = 1, N$ since $F_U(\tau)$ is finite by Lemma 9. Thus $\tau(x)x^{-1} = \tau(y)y^{-1}$ for some y in N. Now $y^{-1}x$ is in $F_G(\tau)$. Since $F_G(\tau) \subseteq T$, we have $T^x = T^{y^{-1}x} = T$. Thus T is the only τ-stable maximal torus of G. It follows that T is the only σ-stable maximal torus of G (this last fact is proved more directly by replacing τ by σ and N by the normalizer of T in G in the above argument).

THEOREM 11. *Suppose that G is not nilpotent and let σ be of finite squarefree order. Then there exist σ_i in $\langle \sigma \rangle$ $(i = 1, 2)$ such that $\sigma = \sigma_1 \sigma_2$, $F_G(\sigma_1)$ contains a torus of positive dimension, and $F_G(\sigma_2)$ contains a closed connected unipotent subgroup of positive dimension.*

PROOF. This is trivial if $F_G(\sigma)$ is infinite (since an infinite linear algebraic group contains either a torus of positive dimension or a closed connected unipotent subgroup of positive dimension, and a connected nonnilpotent algebraic group contains both) [2, Exposé 6].

Now let $F_G(\sigma)$ be finite. Then G is solvable and $G = TU$ where T is a σ-stable torus and U is a closed connected characteristic unipotent subgroup of G (Theorems 7 and 10). Define $U_1 = \{u \in U \mid u$ is central in U and $u^p = 1\}$, $\cdots$, $U_{i+1} = \{u \in U \mid uU_i$ is central in U/U_i and $u^p \in U_i\}$ (at characteristic 0 this is just the ascending central series). For some n, $U = U_n$ (since U is nilpotent by [5], and of finite exponent p^e for some e if the characteristic is $p > 0$). $\langle \sigma \rangle T$ (semidirect) acts in $W_i = U_{i+1}/U_i$ for all $i \geq 1$. Since G is not nilpotent, we can choose i such that T acts nontrivially in W_i (for otherwise $1 \leq U_1 \leq \cdots \leq U_n \leq U_n T = G$ would be a central series for G). Choose such an i and let $W = W_i$. Let $T_k = \{t \in T \mid t^k = 1\}$ $(k$ a

positive integer). Since $\bigcup_{(k,|\sigma|p)=1} T_k$ is dense in T, we can choose k such that $(k, |\sigma| p) = 1$ and T_k acts nontrivially in W. Let $\mathfrak{G} = \langle \sigma \rangle T_k$. W is a vector space (over the field of p elements if the characteristic is $p > 0$; otherwise over the underlying field of G) and $\mathfrak{G}$ acts in W as a group of linear transformations. Since $\mathfrak{G}$ is finite, W has a finite dimensional subspace V, stable under $\mathfrak{G}$, on which T_k acts nontrivially.

We now use a refinement of a technique used by Higman in [4]. Let $\overline{V}$ be the vector space obtained from V by extending the ground field of V to its algebraic closure. If α is a function from T_k into the underlying field of $\overline{V}$, let $\overline{V}_\alpha = \{v \in \overline{V} \mid vt = \alpha(t)v \text{ for all } t \text{ in } T_k\}$. Then $\overline{V} = \Sigma \oplus \overline{V}_\alpha$. Let $R = \{\alpha \mid \overline{V}_\alpha \neq \{0\}\}$. Then $\mathfrak{G}$ acts in R in a natural way and R contains an element α such that $\alpha(t) \neq 1$ for some t in T_k. Choose such an α. Let $\langle \sigma_1 \rangle$ be the isotropy subgroup of $\langle \sigma \rangle$ acting in R at the point α. Since $|\sigma|$ is squarefree, $\langle \sigma \rangle = \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle$ (direct) for some σ_2 in $\langle \sigma \rangle$. σ_1 keeps $\overline{V}_\alpha$ stable whereas $\langle \sigma_2 \rangle$ acts simply transitively on the orbit of α under $\langle \sigma \rangle$; and $(\sigma_2)^i$ maps $\overline{V}_\alpha$ onto $\overline{V}_{\alpha_i}$ where α_i is the image of α under $(\sigma_2)^i$. Let $a = |\sigma_1|$, $b = |\sigma_2|$. Choose t in T_k such that $\alpha(t) \neq 1$ and let $\bar{t} = \sigma_1(t)(\sigma_1)^2(t) \cdots (\sigma_1)^a(t)$. Choose a nonzero element x in $\overline{V}_\alpha$ and let $\bar{x} = \sigma_2(x) + (\sigma_2)^2(x) + \cdots + (\sigma_2)^b(x)$. Then $\sigma_1(\bar{t}) = \bar{t}$ and $\sigma_2(\bar{x}) = \bar{x}$. Now $\bar{t} \neq 1$ since $t | \overline{V}_\alpha$ and $\sigma_1 | \overline{V}_\alpha$ are commuting transformations of $\overline{V}_\alpha$, so that

$$\bar{t} \mid \overline{V}_\alpha = ((\sigma_1)^{-1} t \sigma_1)((\sigma_1)^{-2} t (\sigma_1)^2) \cdots ((\sigma_1)^{-a} t (\sigma_1)^a) \mid \overline{V}_\alpha$$
$$= t^a \mid \overline{V}_\alpha = (\alpha(t))^a \operatorname{id}_{\overline{V}_\alpha} \neq \operatorname{id}_{\overline{V}_\alpha}$$

(since $(a, |T_k|) = 1$ implies that $(\alpha(t))^a \neq 1$). And $\bar{x} \neq 0$ since the elements $\sigma_2(x), \cdots, (\sigma_2)^b(x)$ are linearly independent ($\Sigma \overline{V}_{\alpha_i}$ is direct and $(\sigma_2)^i(x)$ lies in $\overline{V}_{\alpha_i}$ for all i).

Thus $F_{T_k}(\sigma_1) \neq \{1\}$ and $F_{U_{i+1}/U_i}(\sigma_2) \neq \{1\}$. It follows that $F_G(\sigma_1)$ contains a torus of positive dimension (for otherwise $F_T(\sigma_1)$ would be finite, and by Lemma 6, $x^a = 1$ for x in $F_T(\sigma_1)$; but this is impossible since $(a, |T_k|) = 1$ and $F_T(\sigma_1) \geqq F_{T_k}(\sigma_1) > \{1\}$). We next show that $F_G(\sigma_2)$ contains a closed connected unipotent subgroup of positive dimension. For this, let $\sigma_2 = \tau\mu$ where τ, μ are the p-regular and p-singular components of σ_2 respectively. Then $F_{U_{i+1}/U_i}(\tau) \neq \{1\}$. Since $H^1(\langle \tau \rangle, U_i) = \{0\}$ (the exponents of $\langle \tau \rangle$ and U_i are relatively prime), it follows that $F_U(\tau) \neq \{1\}$. Thus $F_U(\tau)$ is infinite by Lemma 6. Since $F_U(\tau)$ is μ-stable, an application of Lemma 3 with $\mathfrak{G} = \langle \mu \rangle F_U(\tau)$ and $\mathfrak{H} = F_U(\tau)$ shows that $F_U(\sigma_2) = F_U(\tau\mu) = F_\mathfrak{G}(\mu)$ is infinite. Thus $F_G(\sigma_2)$ contains a closed connected unipotent subgroup of positive dimension.

THEOREM 12. *If σ has prime period and $F_G(\sigma)$ is finite, then G is nilpotent.*

PROOF. This is immediate from Theorem 11.

We conclude with an example of some of the structure described above (which shows in particular that in Theorem 12 "prime period" cannot be replaced by "prime power period"). Let V be a vector space over F with basis $e_1, \cdots, e_m, f$. Let $g(a, b)$ be the linear transformation of V defined by $g(a, b)e_i = a_i e_i + b_i f$, $g(a, b)f = f$ for $a = (a_1, \cdots, a_m)$, $b = (b_1, \cdots, b_m)$ in F^m. Let $G = \{g(a, b) \mid \pi a_i = 1\}$. G is a group since $g(1, 0) = I$ and $g(a, b)g(c, d) = g(ac, bc + d)$ (where $0 = (0, \cdots, 0)$, $1 = (1, \cdots, 1)$ and xy, $x + y$ are defined componentwise for x, y in F^m). Let $T = \{g(a, 0) \mid \pi a_i = 1\}$, $U = \{g(1, b) \mid b \in F^m\}$. $G = TU$ (semidirect); and G is an algebraic group with maximal torus T and nilradical U. If $m \geq 2$, G is not nilpotent (since T is not central, see [2, §6-04]). The basis change $(e_1, \cdots, e_m, f) \to (\epsilon e_m, e_1, \cdots, e_{m-1}, f)$ (ϵ in F) induces a birational automorphism σ_ϵ of G sending $g(a, b)$ into $g(a', b^\epsilon)$ where $(a_1, \cdots, a_m)' = (a_m, a_1, \cdots, a_{m-1})$ and $(b_1, \cdots, b_m)^\epsilon = (\epsilon b_m, b_1, \cdots, b_{m-1})$. If $\epsilon \neq 1$, $F_G(\sigma_\epsilon) = \{g(a, 0) \mid a = (\delta, \cdots, \delta)$ and $\delta^m = 1\}$ is a finite subset of T. If ϵ is a primitive qth root of 1, σ_ϵ has finite order mq. By varying m and q, one finds various directions in which Theorems 11 and 12 cannot be generalized.

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