GROTHENDIECK GROUPS AND DIVISOR GROUPS[1]

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0. Introduction. Before stating the results in this note, it is necessary to introduce some notation. A is a noetherian integral domain which is integrally closed in its quotient field K. Σ is a central simple finite-dimensional K-algebra, D is a central division K-algebra and V is a finitely generated right D vector space such that $\Sigma = \mathrm{Hom}_D(V, V)$ (so also $D = \mathrm{Hom}_\Sigma(V, V)$).

Let Λ be an A-order in Σ. $\mathfrak{M}(\Lambda)$ denotes the category of left finitely generated Λ-modules, $\mathfrak{J}(\Lambda)$ the Serre subcategory of $\mathfrak{M}(\Lambda)$ consisting of A-torsion left Λ-modules. $\mathcal{P}(\Lambda)$ is the Serre subcategory of $\mathfrak{J}(\Lambda)$ consisting of the *pseudo-nul* left Λ-modules, where a pseudo-nul module M is one for which $M_\mathfrak{p} = A_\mathfrak{p} \otimes_A M = 0$ for all prime ideals $\mathfrak{p}$ of A of height at most one. The category $\mathfrak{M}/\mathcal{P}(\Lambda)$ is formed by taking as objects the objects of $\mathfrak{M}(\Lambda)$ and for M, N in $\mathfrak{M}(\Lambda)$, defining $\mathrm{Hom}_{\mathfrak{M}/\mathcal{P}}(M, N)$ to be the direct limit of $\mathrm{Hom}_{\mathfrak{M}}(M', N')$ taken over those M' and N' such that M/M' is in $\mathcal{P}$ and $N' = N/N''$ with N'' in $\mathcal{P}$. $\mathfrak{J}/\mathcal{P}(\Lambda)$ is formed in a similar fashion. The first result may now be stated as follows:

THEOREM 1. *Let A, Σ, D be as above. Let Λ_1 and Λ_2 be maximal orders in Σ, and Γ a maximal order in D. Then there are functors*

$$F(\Lambda_1, \Lambda_2): \mathfrak{M}(\Lambda_1) \to \mathfrak{M}(\Lambda_2),$$
$$G(\Lambda_2, \Gamma): \mathfrak{M}(\Lambda_2) \to \mathfrak{M}(\Gamma),$$

which induce isomorphisms of the categories

$$\mathfrak{M}/\mathcal{P}(\Lambda_1) \to \mathfrak{M}/\mathcal{P}(\Lambda_2) \to \mathfrak{M}/\mathcal{P}(\Gamma),$$
$$\mathfrak{J}/\mathcal{P}(\Lambda_1) \to \mathfrak{J}/\mathcal{P}(\Lambda_2) \to \mathfrak{J}/\mathcal{P}(\Gamma).$$

If $\mathcal{C}$ is an abelian category, $K^0(\mathcal{C})$ denotes the Grothendieck group of $\mathcal{C}$. It can be defined as follows: For each C in $\mathcal{C}$ there is an $f(C)$ in $K^0(\mathcal{C})$, an abelian group, such that if $0 \to C' \to C \to C'' \to 0$ is an exact sequence in $\mathcal{C}$, then $f(C) = f(C') + f(C'')$. Furthermore, if G is any abelian group and for each $\mathcal{C}$ in C there is a $g(C)$ in G such that $g(C) = g(C') + g(C'')$ on exact sequences in $\mathcal{C}$ then there is a unique homomorphism $h: K^0(\mathcal{C}) \to G$ such that $g = hf$. Let $G_t(\Lambda) = K^0(\mathfrak{J}/\mathcal{P}(\Lambda))$ and $G(\Lambda) = K^0(\mathfrak{M}/\mathcal{P}(\Lambda))$. An immediate corollary

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to Theorem 1 is

COROLLARY. *The functors F and G induce isomorphisms*

$$G_t(\Lambda_1) \to G_t(\Lambda_2) \to G_t(\Gamma),$$
$$G(\Lambda)_1 \to G(\Lambda_2) \to G(\Gamma).$$

In case A is a Dedekind domain these results are known, so in a sense Theorem 1 may be considered to be a generalization of the Morita Theorems which give these isomorphisms in this case (see [5]).

If M is an A-lattice in Σ, define $M^{-1} = \{x \in \Sigma: MxM \subseteq M\}$. Let Λ be a maximal order in Σ. Let $I(\Lambda)$ denote the set of A-lattices in Σ which are both left and right A-modules. Goldman in [6] defined $D(\Lambda)$, the *group of divisors* of Λ, to be the abelian group obtained from $I(\Lambda)$ by the equivalence relation (quasi-equality for two-sided fractionary Λ-ideals).

$$\text{“}M \sim N \text{ in } I(\Lambda) \quad \text{iff} \quad M^{-1} = N^{-1}.\text{”}$$

Thus $D(\Lambda) = I(\Lambda)/\sim$, with multiplication given by $(M, N) \to \overline{M}\overline{N}$. Goldman proves that $D(\Lambda_1)$ is naturally isomorphic to $D(\Lambda_2)$ when Λ_1 and Λ_2 are maximal orders in Σ. The second result of this note is

THEOREM 2. $D(\Lambda)$ *is isomorphic to* $G_t(\Lambda)$.

Theorem 2 and the corollary to Theorem 1 yield the important, but not surprising, result, namely the

COROLLARY. *If* Λ *is a maximal order in* Σ, *and* Γ *a maximal order in* D, *then* $D(\Lambda)$ *is (naturally) isomorphic to* $D(\Gamma)$.

Thus considerations of $D(\Lambda)$ are reduced to considerations of $D(\Gamma)$, but Γ is in a division algebra.

1. **Proof of Theorem 1.** The notations of §0 are retained here. Let Λ and Ω be maximal A-orders in Σ. The *conductor*, $\{x \in \Sigma: \Omega x \subseteq \Lambda\}$, is denoted by $\Lambda:\Omega$. It is an A-lattice in Σ which is a right ideal in Λ and a left Ω-module. Define $F(\Lambda, \Omega): \mathfrak{M}(\Lambda) \to \mathfrak{M}(\Omega)$ by $F(\Lambda, \Omega)(M) = \Lambda:\Omega \otimes_\Lambda M$ for the left Λ-module M. Certainly $F(\Lambda, \Omega)$ is a functor. Since $A_\mathfrak{p}$ is a flat A-module for each prime ideal $\mathfrak{p}$ of A, it is clear that $A_\mathfrak{p} \otimes_A F(\Lambda, \Omega) = F(\Lambda_\mathfrak{p}, \Omega_\mathfrak{p})$ for each prime ideal $\mathfrak{p}$ of A. Hence F takes torsion modules to torsion modules, and pseudo-nul modules to pseudo-nul modules, and consequently induces functors

$$F'(\Lambda, \Omega): \ \mathfrak{M}/\mathcal{P}(\Lambda) \to \mathfrak{M}/\mathcal{P}(\Omega),$$
$$F''(\Lambda, \Omega): \ \mathfrak{I}/\mathcal{P}(\Lambda) \to \mathfrak{I}/\mathcal{P}(\Omega).$$

(F'' is induced by F'.)

To show that F' (and hence F'') is an isomorphism, it is sufficient to construct a functorial inverse. But, consider the natural transformation

$$F(\Omega, \Lambda)F(\Lambda, \Omega) \to I_{\mathfrak{M}(\Lambda)}$$

given by $(\Omega:\Lambda) \otimes_{\Omega} (\Lambda:\Omega) \otimes_{\Lambda} M \to M: \omega\otimes\lambda\otimes m \to \omega\lambda m$. Upon localizing at a height one or less prime ideal of A, one obtains an identification; that is, $F(\Omega_{\mathfrak{p}}, \Lambda_{\mathfrak{p}})F(\Lambda_{\mathfrak{p}}, \Omega_{\mathfrak{p}}) = I$. For in case $\mathfrak{p}=0$, $\Omega_{\mathfrak{p}} = \Sigma = \Lambda_{\mathfrak{p}}$, and in the other cases, $A_{\mathfrak{p}}$ is a discrete rank-one valuation ring, so $\Lambda_{\mathfrak{p}}: \Omega_{\mathfrak{p}} = u\Lambda_{\mathfrak{p}} = \Omega_{\mathfrak{p}}u$ and $\Omega_{\mathfrak{p}}: \Lambda_{\mathfrak{p}} = u^{-1}\Omega_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}u^{-1}$, where u is a unit in Σ (by 3.4 of [**1**]). Hence F' (and so F'') is an isomorphism.

Using the same arguments, one shows that $F'(\Lambda, \Omega)F'(\Omega, \Omega') = F'(\Lambda, \Omega')$ for maximal A-orders in Σ. This says that the isomorphisms are natural.

Before proving the second part of Theorem 1, a generalization of Proposition 4.2 of [**1**] is needed.

The proof is exactly as in [**1**]. Proposition 4.1 of [**1**] and its proof remain valid when Hom is replaced by Hom_{Γ} and $\otimes$ by $\otimes_{\Gamma}$, so it can be used as in the proof of [**1**, Proposition 4.2].

PROPOSITION 1. *Let A be a noetherian integrally closed integral domain with quotient field K. Let Σ be a finite-dimensional central simple K-algebra. Suppose $\Sigma = \mathrm{Hom}_D(V, V)$ where D is a central division K-algebra and V a finite-dimensional right D-module. An A-order Λ in Σ is maximal if, and only if, there is a maximal A-order Γ in D and a right Γ-submodule E of V which is a reflexive A-lattice such that $\Lambda = \mathrm{Hom}_{\Gamma}(E, E)$. In this case $\Gamma = \mathrm{Hom}_{\Lambda}(E, E)$.*

Let Λ be a maximal order in Σ and let E and Γ be as in Proposition 1. Define $G(\Lambda, \Gamma): \mathfrak{M}(\Lambda) \to \mathfrak{M}(\Gamma)$ by $G(\Lambda, \Gamma)(M) = \mathrm{Hom}_{\Gamma}(E, \Gamma) \otimes_{\Lambda} M$. The localization arguments used above show that $G(\Lambda, \Gamma)$ preserves torsion and pseudo-nullity, so G induces

$$G'(\Lambda, \Gamma): \mathfrak{M}/\mathcal{P}(\Lambda) \to \mathfrak{M}/\mathcal{P}(\Gamma),$$
$$G''(\Lambda, \Gamma): \mathfrak{I}/\mathcal{P}(\Lambda) \to \mathfrak{I}/\mathcal{P}(\Gamma).$$

There is also the functor $G(\Gamma, \Lambda): \mathfrak{M}(\Gamma) \to \mathfrak{M}(\Lambda)$ defined by $G(\Gamma, \Lambda)(N) = E \otimes_{\Gamma} N$. As before, there are natural transformations

$$G(\Lambda, \Gamma)G(\Gamma, \Lambda) \to I_{\mathfrak{M}(\Gamma)},$$
$$G(\Gamma, \Lambda)G(\Lambda, \Gamma) \to I_{\mathfrak{M}(\Lambda)}.$$

The first is given by the natural homomorphism $\mathrm{Hom}_{\Gamma}(E, \Gamma) \otimes_{\Lambda} E \to \mathrm{Hom}_{\Lambda}(E, E) = \Gamma$, the second by $E \otimes_{\Gamma} \mathrm{Hom}_{\Gamma}(E, \Gamma) \to \mathrm{Hom}_{\Gamma}(E, E) = \Lambda$ (cf. [**1**, Proposition A.4]). Once again, these localize to identifications so

$$G'(\Lambda, \Gamma)G'(\Gamma, \Lambda) = I_{\mathfrak{M}/\mathcal{P}(\Gamma)};$$
$$G'(\Gamma, \Lambda)G'(\Lambda, \Gamma) = I_{\mathfrak{M}/\mathcal{P}(\Lambda)}.$$

This concludes the proof of Theorem 1.

Heller and Reiner in [4], [5] discuss the exact sequences

$$\text{(HR)}\qquad \begin{array}{l} K^1(\Sigma) \to G_t(\Lambda) \to G(\Lambda) \to K^0(\Sigma) \to 0, \\ K^1(D) \to G_t(\Gamma) \to G(\Gamma) \to K^0(D) \to 0, \end{array}$$

where A is a Dedekind domain.

The corollary to Theorem 1 generalizes the discussion on pp. 351–352 of [5], i.e. it implies that these are isomorphic sequences for any noetherian integrally closed integral domain A.

Another application of the corollary to Theorem 1 is

Proposition 2. *Let A be a noetherian integrally closed integral domain with quotient field K. Let V be a finite-dimensional vector space over K and let $\Sigma = \mathrm{Hom}_K(V, V)$. Let Λ be a maximal order in Σ. Then*

$$G_t(\Lambda) = D(A) \qquad (\textit{divisor group of } A),$$
$$G(\Lambda) = C(A) \oplus \boldsymbol{Z}\ (C(A) = \textit{class group of } A).$$

Proof. By the corollary to Theorem 1, $G_t(\Lambda) = G_t(A)$ and $G(\Lambda) = G(A)$. By Proposition 11 of [3, §4, n°5], $G_t(A) = D(A)$. By Proposition 17 of [3, §4, n°8], $G(A) = C(A) \oplus \boldsymbol{Z}$.

Remark. Theorem 2 is a generalization of this proposition.

2. **Proof of Theorem 2.** The proof of the theorem is exactly the proof of Proposition 11 of [3, §4, n°5] modified to the present situation.

Let Λ be a maximal A-order in Σ. For each prime (two-sided) ideal $\mathfrak{P}$ of Λ of height one let div $\mathfrak{P}$ denote its image in $D(\Lambda)$. In [7] it is proved that there is a bijection, given by $\mathfrak{P} \to \mathfrak{P} \cap A$, of the set of prime ideals of height one of Λ to the set of prime ideals of height one of A. Let $P(\Lambda)$ denote the set of prime ideals of Λ.

Let $M \in \mathfrak{I}(\Lambda)$. Then if $\mathfrak{p}$ is a prime ideal of A, the $\Lambda_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ has finite length, denoted by $l_{\mathfrak{p}}(M_{\mathfrak{p}})$. Since $M_{\mathfrak{p}} = 0$ if $M \in \mathcal{P}(\Lambda)$, there is induced a map

$$\chi\colon \mathfrak{I}/\mathcal{P}(\Lambda) \to D(\Lambda)$$

defined by $\chi(M) = \sum l_{\mathfrak{p}}(M_{\mathfrak{p}})$ div $\mathfrak{P}$, $\mathfrak{p} = \mathfrak{P} \cap A$, $\mathfrak{P} \in P(A)$. The theorem will be proved if it can be shown that $(D(\Lambda), \chi)$ satisfies the universal mapping property defining the Grothendieck group.

For a Λ-module M, let Ass M denote the set of prime (two-sided) ideals $\mathfrak{P}$ of Λ such that there is a nonzero submodule M' of M with $\operatorname{Ann}_\Lambda M'' = \mathfrak{P}$ for every nonzero submodule M'' of M' (see [7]).

PROPOSITION 3. *Let M be a finitely generated left Λ-module. Then there is a chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_r = 0, r \geqq 0$, such that M_i/M_{i+1} is isomorphic to a module $\Lambda/\mathfrak{N}_i$, $\mathfrak{N}_i$ a left ideal of Λ, where* $\operatorname{Ass} \Lambda/\mathfrak{N}_i = \{\mathfrak{P}_i\}$ *and* $\operatorname{Ann}_\Lambda(\Lambda/\mathfrak{N}_i) = \mathfrak{P}_i$, *$\mathfrak{P}_i$ a prime ideal of Λ.*

The proof is the same as for Theorem 1 of [2, §1, n°4] and is omitted.

It is clear that χ is additive on exact sequences, so Proposition 3 shows that

$$\chi(M) = \sum_{i=0}^{r-1} \chi(\Lambda/\mathfrak{N}_i)$$

where the $\mathfrak{N}_i$ are left ideals satisfying the conclusion of Proposition 3.

The next proposition permits a study of these modules.

PROPOSITION 4. *Let $\mathfrak{P} \in P(\Lambda)$, $\mathfrak{p} = \mathfrak{P} \cap A$. Let $\mathfrak{m}$ be a minimal left ideal in the simple $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$-algebra $\Lambda_\mathfrak{p}/\mathfrak{P}\Lambda_\mathfrak{p}$. Let $\mathfrak{n} = \mathfrak{m} \cap (\Lambda/\mathfrak{P})$. Then*

(i) *If $\mathfrak{N}$ is a left ideal of Λ such that* $\operatorname{Ass} \Lambda/\mathfrak{N} = \{\mathfrak{P}\}$ *and $\mathfrak{N} \supseteq \mathfrak{P}$, then the class of $\Lambda/\mathfrak{N}$ in $G_t(\Lambda)$ is some integral multiple of the class of $\mathfrak{n}$ in $G_t(\Lambda)$.*

(ii) $\chi(\mathfrak{n}) = \operatorname{div} \mathfrak{P}$.

PROOF. Throughout this proof let $S = \Lambda/\mathfrak{P}$. Let $[M]$ denote the class of M in $G_t(\Lambda)$.

Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two minimal left ideals in $S_\mathfrak{p}$. Then there is a t in S, t a unit in $S_\mathfrak{p}$, such that $\mathfrak{m}_2 = \mathfrak{m}_1 t$. Let $\mathfrak{n}_i = \mathfrak{m}_i \cap S$. Then $\mathfrak{n}_1 t \subseteq \mathfrak{n}_2$, so consider the homomorphism $\mathfrak{n}_1 \xrightarrow{t} \mathfrak{n}_2$. When localized at $\mathfrak{p}$ it is the isomorphism $\mathfrak{m}_1 \xrightarrow{t} \mathfrak{m}_2$. If $\mathfrak{q}$ is a prime ideal of height one of A distinct from $\mathfrak{p}$, then $(\mathfrak{n}_1)_\mathfrak{q} = 0 = (\mathfrak{n}_2)_\mathfrak{q}$, so t localized at $\mathfrak{q}$ is also an isomorphism. So in $\mathfrak{I}/\mathcal{P}(\Lambda)$ this map is an isomorphism, hence $[\mathfrak{n}_1] = [\mathfrak{n}_2]$.

Suppose that $\mathfrak{N}$ is a left ideal satisfying the hypotheses of condition (i). Then $(\mathfrak{N}/\mathfrak{P})_\mathfrak{p}$ is a left ideal in $S_\mathfrak{p}$, so is the direct sum of minimal left ideals $\mathfrak{m}_1, \cdots, \mathfrak{m}_t$ of $S_\mathfrak{p}$. Let $\mathfrak{n}_i = S \cap \mathfrak{m}_i$ and consider $\mathfrak{n}_1 + \cdots + \mathfrak{n}_t$ in S. This sum is direct. The homomorphisms $\mathfrak{n}_1 + \cdots + \mathfrak{n}_t \to (\mathfrak{N}/\mathfrak{P})_\mathfrak{p} \cap S$ and $\mathfrak{N}/\mathfrak{P} \to (\mathfrak{N}/\mathfrak{P})_\mathfrak{p} \cap S$ are isomorphisms at every localization. Hence $t[\mathfrak{n}] = [\mathfrak{n}_1 + \cdots + \mathfrak{n}_t] = [\mathfrak{N}/\mathfrak{P}]$. This holds when $\mathfrak{N} = \Lambda$, so let $[\Lambda/\mathfrak{P}] = [\mathfrak{n}_1 + \cdots + \mathfrak{n}_s] = s[\mathfrak{n}]$ where $s = [(\Lambda/\mathfrak{P})_\mathfrak{p} : (A/\mathfrak{p})_\mathfrak{p}]$. Then $t \leqq s$.

Now consider the exact sequence $0 \to \mathfrak{N}/\mathfrak{P} \to \Lambda/\mathfrak{P} \to \Lambda/\mathfrak{N} \to 0$. Then

$$[\Lambda/\mathfrak{N}] = [\Lambda/\mathfrak{P}] - [\mathfrak{N}/\mathfrak{P}]$$
$$= s[\mathfrak{n}] - t[\mathfrak{n}]$$
$$= (s - t)[\mathfrak{n}].$$

So (i) has been established. (ii) is clear from the definition of $\mathfrak{n}$.

COROLLARY. *For each* $\mathfrak{P}\in P(\Lambda)$, *let* $\mathfrak{n}(\mathfrak{P})$ *be a module constructed in Proposition* 4. *Then* $G_t(\Lambda)$ *is free on the set* $[\mathfrak{n}(\mathfrak{P})]$.

This follows immediately from the two previous propositions.

PROPOSITION 5. *For each torsion left* Λ*-module* M, *let* $g(M)$ *be an element in an abelian group* G. *Suppose* g *satisfies the two conditions*

a. *If* $0\to M'\to M\to M''\to 0$ *is an exact sequence in* $\mathfrak{J}(\Lambda)$, *then* $g(M) = g(M')+g(M'')$.

b. *If* $M\in\mathcal{P}(\Lambda)$, *then* $g(M)=0$.

Then there is a unique homomorphism $\theta\colon D(\Lambda)\to G$ *such that* $g=\theta\chi$.

PROOF. Let $\mathfrak{n}(\mathfrak{P})$ be an ideal of $\Lambda/\mathfrak{P}$ defined in Proposition 4. Let $\theta(\operatorname{div}\mathfrak{P})=g(\mathfrak{n}(\mathfrak{P}))$. Then continue as in Proposition 11 of [**3**, §4, n°5]. Propositions 3 and 4 are designed to make that proof work.

Proposition 5 shows that $D(\Lambda)$ satisfies the universal property which defines the Grothendieck group, so it must be isomorphic to it. This completes the proof of Theorem 2.

REMARK. Since $K^0(\Sigma)=K^0(D)=\boldsymbol{Z}$ in (HR) and $\boldsymbol{Z}$ is $\boldsymbol{Z}$ projective, $G^0(\Gamma)=C(\Gamma)\oplus\boldsymbol{Z}$ where $C(\Gamma)$ is the kernel of $G^0(\Gamma)\to K^0(D)$, and hence is the image of $G_t^0(\Gamma)\to G^0(\Gamma)$. A natural question is: What is an ideal (or module) theoretical description of the subgroup H of $D(\Gamma)$ such that $D(\Gamma)/H=C(\Gamma)$? $C(\Gamma)$ is a generalization of the commutative class group (see [**3**]). A corollary to the corollary to Theorem 1 is that $C(\Lambda)$ is isomorphic to $C(\Gamma)$ and both do not depend on the maximal orders in question.

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