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THE ASYMPTOTIC BEHAVIOR OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS[1]

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We shall study the asymptotic behavior for $t \to \infty$ of solutions of the following nonlinear differential equation:

$$u'' + f(t, u) = 0. \tag{1}$$

We suppose that $f(t, u)$ satisfies the following conditions:

H-1: $f(t, u)$ is continuous in D: $t \geqq 0$, $-\infty < u < \infty$.

H-2: The derivative f_u exists on D and satisfies $f_u(t, u) > 0$ on D.

H-3: $|f(t, u(t))| \leqq f_u(t, 0)|u(t)|$ on D.

An important class of functions $f(t, u)$ which satisfy conditions H-1, 2, 3 is the class of twice continuously differentiable functions $f(t, u)$ which are odd and strictly monotone in u with $f_{uu} \geqq 0$ for $u < 0$ and $f_{uu} \leqq 0$ for $u > 0$. Nonlinear eigenvalue problems involving this class of functions have been studied extensively by G. H. Pimbley [1].

For the case $f(t, u) = \pm t^\sigma u^n$, R. Bellman [2] has given an exhaustive treatment of the asymptotic behavior of proper solutions (i.e., solutions which exist and have continuous derivatives for $t \geqq t_0$). For the case $f(t, u) = a(t)u^{2n+1}$ several results on asymptotic behavior exist depending on properties of $a(t)$. References can be found in the papers of P. Waltman [3] and R. A. Moore and Z. Nehari [4].

Our basic result is that there exist solutions of (1) which approach those of $u'' = 0$. More precisely, we prove the

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THEOREM. *Let $f(t, u)$ satisfy* H-1, 2, 3, *and in addition, suppose that*

$$\int^{\infty} t f_u(t, 0)\,dt < \infty. \tag{2}$$

Then, equation (1) *has solutions which are asymptotic to $a+bt$ as $t\to\infty$, where $b\neq 0$.*

PROOF. Our proof is essentially a modification of one given by Bellman [2, pp. 114–115] for the linear case $f(t, u)=a(t)u$. Integrate (1) twice between 1 and t to obtain

$$u(t) = c_1 + c_2 t - \int_1^t (t-s) f(s, u(s))\,ds. \tag{3}$$

From this we obtain, for $t\geqq 1$,

$$|u(t)| \leqq (|c_1| + |c_2|)t + t\int_1^t |f(s, u(s))|\,ds.$$

Using properties H-2, 3, we then obtain

$$\frac{|u(t)|}{t} \leqq (|c_1| + |c_2|) + \int_1^t s f_u(s, 0)\frac{|u(s)|}{s}\,ds. \tag{4}$$

We now invoke the fundamental Gronwall inequality which states that if $u, v\geqq 0$, $c>0$ and if $u(x)\leqq c+\int_0^x u(\xi)v(\xi)d\xi$, then

$$u(x) \leqq c\exp\left(\int_0^x v(\xi)\,d\xi\right).$$

Applying this to (4), we obtain

$$\frac{|u(t)|}{t} \leqq (|c_1| + |c_2|)\exp\left(\int_1^t s f_u(s, 0)\,ds\right). \tag{5}$$

Finally, (2) and (5) imply

$$|u(t)|/t \leqq c_3. \tag{6}$$

Now, by differentiating (3), we obtain

$$u'(t) = c_2 - \int_1^t f(s, u(s))\,ds. \tag{7}$$

Then, H-3 and (6) imply that

$$\int_1^t |f(s, u(s))|\,ds \leqq \int_1^t f_u(s, 0)\,|u(s)|\,ds \leqq c_3\int_1^t s f_u(s, 0)\,ds.$$

Hence, as $t \to \infty$, the integral in (7) converges, and therefore u' has a limit as $t \to \infty$. To ensure that this limit is not zero, we choose $c_2 = 1$ and use as a lower limit, instead of 1, a point t_0, where t_0 is chosen so that $c_3 \int_{t_0}^{\infty} s f_u(s, 0) ds < 1$. Q.E.D.

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