FOURIER COEFFICIENTS OF FUNCTIONS OF BOUNDED VARIATION

MITCHELL TAIBLESON[1]

The fact that $\int_{2\pi k/|n|}^{2\pi(k+1)/|n|} e^{-inx}dx=0$ will be used to prove that the Fourier coefficients of a function of bounded variations are $O(1/n)$.

If f is of bounded variation on $[a, b]$, let $T(a, b)$ be the total variation of f on $[a, b]$.

THEOREM. *If f is of bounded variation on $[0, 2\pi]$, $f(x)\sim \sum c_n e^{inx}$, then $c_n=O(1/n)$ as $|n|\to\infty$.*

PROOF. We may suppose that $n\neq 0$. Fix n and let $a_k=2k\pi/|n|$, $k=0, 1, \cdots, |n|$. Let g be a step function equal to $f(a_k)$ on (a_{k-1}, a_k), $k=1, \cdots, |n|$. Then $\int_0^{2\pi} g(x)e^{-inx}dx=0$ and $c_n=\int_0^{2\pi} f(x)e^{-inx}dx$. Hence

$$\begin{aligned} |c_n| &= \left|\int_0^{2\pi}(f(x)-g(x))e^{-inx}dx\right| \\ &\leqq \sum_{k=1}^{|n|}\int_{a_{k-1}}^{a_k}|f(x)-f(a_k)|\,dx \\ &\leqq \sum_{k=1}^{|n|} T(a_{k-1}, a_k)2\pi/|n| \\ &= T(0, 2\pi)2\pi/|n|. \end{aligned}$$

Q.E.D.

WASHINGTON UNIVERSITY AND
INSTITUTE FOR ADVANCED STUDY

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