SCHWARTZ DISTRIBUTIONS ANALYTIC IN A PARAMETER AND ASSOCIATED SEMI-ANALYTIC FUNCTIONS

CLIFFORD A. LONG[1]

1. **Introduction.** The notion of distributions depending on a parameter was introduced by Laurent Schwartz and is indicated in the pamphlet on his Canadian lectures written by Israel Halperin [**3**, p. 17]. Gel'fand and Šilov [**2**, p. 147] studied the properties of generalized functions depending on a parameter, particularly when the dependence is analytic in the sense indicated below. We show that restrictions placed on distributions over a region Z imply certain *associated* functions; in particular (Theorem 3), that analytic distributions have associated semi-analytic functions.

2. **Basic definitions and results.** Let $F(I)$ denote a continuous linear functional on a space K_I of testing functions ϕ with support contained in an open interval I, and let $F(\phi)$ be the corresponding real numbers. The properties of distributions depending on a parameter are in many ways analogous to those of ordinary functions through use of the basic completeness theorem of Schwartz [**4**, p. 74]: If a sequence of distributions F_i is such that, for each ϕ in K, $F_i(\phi)\rightarrow F(\phi)$, then these numbers $F(\phi)$ determine a distribution F.

If z is a complex parameter taking on values in a region Z and for each z there exists a corresponding distribution $F(I; z)$ with support in I, we call $F(I; z)$ a distribution on I over Z. $F_0(I)$ is the limit of $F(I; z)$ as $z\rightarrow z_0$ iff the ordinary function values $F(\phi, z)$ approach $F_0(\phi)$ for every ϕ in K_I as $z\rightarrow z_0$. *Continuity*, *differentiability* and *analyticity* of $F(I; z)$ with respect to z are defined in the standard way using this limit.

$F(I; z)$ is said to be *identified* with the point function f iff for each closed bounded interval R in I, $F(\phi, z)=\int_R f(x, z)\phi(x)dx$ for each ϕ in K_R and each z in Z. For a closed bounded interval R, $F(R; z)$ is *associated* with f iff there is an integer $r\geqq 0$ such that for each ϕ in K_r and each z in Z, $F(\phi, z)=\int_R f(x, z)\,\phi^{(r)}(x)dx$. If f is continuous in (x, z) and analytic in z throughout $I\times Z$, then f is *semi-analytic* over $I\times Z$. (This definition is due to Maxime Bôcher who demonstrated properties of such functions [**1**].)

Received by the editors March 26, 1965.

[1] The results of this paper are contained in the author's University of Illinois thesis which was supervised by Pierce W. Ketchum. The research was supported in part by the National Science Foundation.

If for each closed bounded interval R in I, $F(R; z)$ is associated with a semi-analytic function, then $F(I; z)$ is an analytic distribution on I over Z. That functions associated with analytic distributions need not be semi-analytic follows from the fact that f is only determined to within an arbitrary polynomial P in x of degree less than r, i.e., $F(\phi, z) = \int_R f(x, z)\phi^{(r)}(x)dx = \int_R [f(x, z) + P(x, z)]\phi^{(r)}(x)dx$, where the coefficients of P may be nonanalytic functions of z.

3. Distributions over Z and associated functions.

THEOREM 1. *If $F(I; z)$ is a continuous distribution over a closed bounded region Z, then for each closed bounded interval R in I, there exists an associated f which is uniformly continuous over $R \times Z$.*

PROOF. (A modification of one given by Schwartz [3, p. 14] which relies on his results.) For each $R = [a, b]$ in I, there exists an associated g bounded on $R \times Z$ such that $F(\phi, z) = \int_R g(x, z)\phi^{(s)}(x)dx$. Integration by parts yields $F(\phi, z) = \int_R h(x, z)\phi^{(s+1)}(x)dx$ where $h(x, z) = -\int_a^x g(t, z)dt$ is continuous in x on R uniformly over Z. Also $F(\phi, z) = \int_R [h(x, z) + P(x, z)]\phi^{(s+1)}(x)dx$, with P an arbitrary polynomial in x of degree less than $s+1$. The following lemma completes the proof of the theorem.

LEMMA 1. *Let $F(I; z)$ be a continuous distribution over closed region Z, and for each closed bounded interval R in I, let there be an associated function, h, continuous in x on R uniformly over Z such that $F(\phi, z) = \int_R h(x, z)\phi^{(r)}(x)dx$. Then there exists a polynomial P in x of degree less than r with bounded functions of z as coefficients, such that: $h(x, z) + P(x, z) = f(x, z)$ defines a bounded function over $R \times Z$, continuous in x on R uniformly over Z, continuous in z on Z uniformly over R, and associated with $F(I; z)$.*

PROOF OUTLINE OF LEMMA 1. The proof is by induction on r. For the case $r = 0$, we need to show that $h(x, z)$ is continuous in z on $\bar{Z}$ uniformly over R, or, what is equivalent, that $g(x, z) = h(x, z) - h(x, z_0)$ converges to 0 uniformly over R as $z \to z_0$, for each z_0 in $\bar{Z}$. For each $\epsilon > 0$, there exists a δ such that, for all z in $\bar{Z}$, $|g(x, z) - g(y, z)| < \epsilon$, if $|x - y| < \delta$. Let the closed bounded interval $R = [a, b]$ be covered by a finite number of intervals $I_1, I_2, \cdots, I_t$, each of length $< \delta$. We now show that there exists a neighborhood N_0 of z_0 in which, for each I_p, there is at least one x_p in I_p such that $g(x_p, z) < \epsilon$ for all z in N_0. If such an N_0 did not exist, then we could find a sequence of nested neighborhoods N_i of z_0 with diameters $\to 0$, such that for each N_i, there exist an $I_p = (c, d)$ and a z_i in N_i with $g(x, z_i) \geqq \epsilon$ for each x

in I_p. For the function with values $\phi_p(x)=\exp(1/(x-d)+1/(c-x))$ on I_p and 0 elsewhere, we would then have $G(\phi_p, z_i)=\int_a^b g(x, z_i)\phi_p(x)dx \geqq \epsilon\int_c^d \phi_p(x)dx=\alpha>0$. This contradicts the fact that $G(\phi, z)=F(\phi, z)-F(\phi, z_0)$ converges to 0. Since there exist only a finite number of I_p's, even if we allowed the I_p to vary for different z_i's, we would still obtain a contradiction. Hence N_0 exists.

Thus, for each x in R and all z in N_0, $|g(x, z)|=|[g(x, z)-g(x_p, z)]-g(x_p, z)|<2\epsilon$ and $g(x, z)$ converges to 0 uniformly on R as $z\to z_0$.

Suppose that the lemma holds for $r=p$ and consider the distribution $F(R; z)$ with $F(\rho, z)=\int_R h(x, z)\rho^{(p-1)}(x)dx$ where h is continuous in x on R uniformly over Z. For each ρ in K_R, and a fixed θ in K_R with $\int_R\theta(x)dx=1$, there exists a function ϕ in K_R, such that $\rho'(x)=\phi(x)-a_0\theta(x)$, where $a_0=\int_R\phi(x)dx$. The identity, $\int_R h(x, z)\rho^{(p+1)}(x)dx=\int_R(h(x, z)+b(z)x^p)\phi^{(p)}(x)dx$, where b is a bounded function of z dependent on θ and on g, but not on ϕ is known [**3**, p. 15]. The form, $G(\phi, z)=\int_R[h(x, z)+b(z)x^p]\phi^{(p)}(x)dx$, determines a distribution on R over Z. It is continuous over Z, because if ϕ is fixed, then $G(\phi, z)=F(\rho, z)$ where $\rho(x)=\int_a^x[\phi(x)-a_0\theta(x)]dx$ and $F(\rho, z)$ is continuous over Z by hypothesis. The case $r=p$ implies the existence of a polynomial Q in x of degree $<p$ with bounded functions of z as coefficients, such that $h(x, z)+b(z)x^p+Q(x, z)$ is bounded over $R\times Z$, continuous in x on R uniformly over Z and continuous in z on Z uniformly over R. The polynomial, P, defined by $P(x, z)=b(z)x^p+Q(x, z)$, satisfies the conditions of the lemma for $r=p+1$.

THEOREM 2. *Let $F(I; z)$ be a continuous distribution over a region Z and its boundary. Then $F(I; z)$ is analytic over Z iff for each closed bounded interval R in I, there exists an associated f continuous on $R\times Z$ such that for each closed contour C in Z, $\int_C f(x, z)dz$ equals a polynomial in x of degree less than r (in general depending on C).*

PROOF (ONLY IF). For C a closed contour in Z, $\int_C F(\phi, z)dz=0$, for each ϕ in K_I by Cauchy's integral theorem. By Theorem 1, for each closed bounded interval R in I, there exists an associated f continuous on $R\times Z$. Hence,

$$\int_C F(\phi, z)dz = \int_C\int_R f(x, z)\phi^{(r)}(x)dxdz = \int_R\int_C f(x, z)dz\phi^{(r)}(x)dx.$$

Since $\int_C f(x, z)dz=g(x)$ describes a continuous function of x, and $\int_R g(x)\phi^{(r)}dx=0$ for all ϕ in K_R, we have that $g(x)$ is a polynomial of degree less than r in x. It may depend on C.

(IF). For each ϕ in K_R and any closed contour C in Z, $\int_C F(\phi, z)dz=\int_C\int_R f(x, z)\phi^{(r)}(x)dxdz=\int_R\int_C f(x, z)dz\phi^{(r)}(x)dx=0$, since $\int_C f(x, z)dz$

is a polynomial in x of degree less than r. By Morera's Theorem, $F(\phi, z)$ is analytic over Z for each ϕ in K_R. Since each ϕ in K_I is in some K_R, $F(\phi, z)$ is analytic over Z for each ϕ in K_I, i.e., $F(I; z)$ is an analytic distribution over Z.

COROLLARY. *If $F(I; z)$ is an analytic distribution over Z identified with the function f continuous over $I\times Z$, then f is semi-analytic over $I\times Z$.*

THEOREM 3. *Let $F(I; z)$ be a distribution over a region Z and its boundary. Then $F(I; z)$ is analytic over region Z and continuous on its boundary iff for each closed bounded interval R in I, there exists an associated semi-analytic function.*

PROOF (IF). Straightforward.

(ONLY IF) Let $F(I; z)$ be analytic over region Z and continuous on its boundary. Then by Theorem 2, for each closed bounded interval there exists an associated function h continuous on $R\times\overline{Z}$. This h satisfies the conditions of Lemma 1. By induction we show that the polynomial P of the lemma can be chosen so that $f=h+P$ will be continuous over $R\times\overline{Z}$ and analytic over Z.

For the case $r=0$, $f=h$, and hence is already continuous over $R\times\overline{Z}$. That f is also analytic over Z follows from the above corollary to Theorem 2.

Suppose that a polynomial P_p exists for the case $r=p$. To find a polynomial for the case $r=p+1$, we use the method from the proof of Lemma 1 to show that

$$\int_R h(x, z)\rho^{(p+1)}(x)dx=\int_R(h(x, z)+b(z)x^p)\phi^{(p)}(x)dx.$$

Hence, the existence of polynomial P_p such that $h(x, z)+b(z)x^p+P_p(x, z)$ defines a semi-analytic function gives us the required polynomial Q with values $b(z)x^p+P_p(x, z)$ for the case $r=p+1$.

Thus, for each closed bounded interval R in I, there exists an associated semi-analytic function $h+P$.

REFERENCES

1. M. Bôcher, *On semi-analytic functions of two variables*, Ann. of Math. (2) **12** (1910–1911), 18–26.

2. I. M. Gel'fand and G. E. Šilov, *Generalized functions*, Vol. 1. *Properties and operations*, Academic Press, New York, 1964. Transl. of 1958 Russian ed.

3. I. Halperin, *Introduction to the theory of distributions*, University of Toronto Press, Toronto, 1952.

4. L. Schwartz, *Theorie des distributions*, Vol. 1, Actualitiés Sci. Ind. No. 1091, Hermann, Paris, 1950.

BOWLING GREEN STATE UNIVERSITY