DIFFERENTIAL SYSTEMS ON FIBERED MANIFOLDS

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Let M and M' be real analytic manifolds and $\rho: M \rightarrow M'$ an analytic map which is surjective and whose rank is equal to the dimension of M' at every point of M. We shall denote respectively by ρ_* and ρ^* the maps induced by ρ on tangent vectors and differential forms.

Let Σ and Σ' be analytic exterior differential systems defined respectively in M and M' and assume that for every differential form $\omega \in \Sigma'$, $\rho^*\omega$ belongs to Σ. Take a point $x_0 \in M$ and put $x_0' = \rho(x_0)$. Let $\mathfrak{V}'^p$ be an integral manifold of Σ' going through x_0'. In this note we give a condition for the existence of an integral manifold $\mathfrak{V}^p$ of Σ going through x_0 such that, for a suitable neighborhood U of x_0, $\rho(\mathfrak{V}^p) = \mathfrak{V}'^p \cap \rho(U)$.

The proof consists in a careful application to our situation of the technic of the Cartan-Kähler theory. The situation we study here appears in the theory of continuous pseudogroups (see [**3**, p. 125]). For the definitions and results we use of the theory of exterior differential systems we refer the reader to [**1**] and [**2**].

For any integral contact element $E^k(x_0)$ of Σ denote by $J(E^k(x_0))$ the polar space of $E^k(x_0)$ and by $J'(E^k(x_0))$ the subspace of $J(E^k(x_0))$ of all forms $\omega \rfloor X_1 \wedge \cdots \wedge X_r$, where ω is a form of degree $r+1$ belonging to $\rho^*(\Sigma')$ and $X_1, \cdots, X_r$ are vectors of $E^k(x_0)$.

Let F_{x_0} be the tangent space to the fiber of M at the point x_0 and denote by $J(E^k(x_0)) \mid F_{x_0}$ the space obtained restricting the forms of $J(E^k(x_0))$ to the subspace F_{x_0}.

Denote by $E'^p(x_0')$ the tangent space of $\mathfrak{V}'^p$ at x_0' and assume that there exists an ordinary integral element $E^p(x_0)$ of Σ such that $\rho_*(E^p(x_0)) = E'^p(x_0)$. Assume moreover there exists a sequence $E^0(x_0) \subset E^1(x_0) \subset \cdots \subset E^{p-1}(x_0)$ of regular contact elements contained in $E^p(x_0)$ and such that $\dim J(E^k(x_0)) - \dim J'(E^k(x_0)) = \dim J(E^k(x_0)) \mid F_{x_0}$, $0 \leqq k \leqq p-1$. Under these assumptions we shall prove the following theorem.

THEOREM. *There exists an integral manifold $\mathfrak{V}^p$ of Σ defined in a neighborhood U of x_0 such that the tangent space $\mathfrak{V}^p_{x_0}$ of $\mathfrak{V}^p$ at the point x_0 is $E^p(x_0)$ and $\rho(\mathfrak{V}^p) = \mathfrak{V}'^p \cap \rho(U)$.*

PROOF. Choose coordinates x^i, $1 \leqq i \leqq n'$ in M', defined in a neighborhood of x_0' and coordinates x^i, y^j, $1 \leqq j \leqq n$ in M, defined in a

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neighborhood of x_0 such that $dx^{k+1}|E^k(x_0)=\cdots=dx^{n'}|E^k(x_0)=dy^j|E^k(x_0)=0$, $1\leqq k\leqq p$, $1\leqq j\leqq n$ (we shall use the same notation for x^i and $x^i\circ\rho$). Since the differentials $dx^1,\cdots,dx^p$ are linearly independent on $E'^p(x_0)$, $\mathcal{V}'^p$ is locally defined by equations $x^\lambda=H^\lambda(x^1,\cdots,x^p)$, $p+1\leqq\lambda\leqq n'$. Since $\mathcal{V}'^p$ is tangent to $E'^p(x_0')$ and $dx^\lambda|E'^p(x_0)=0$, $(\partial H^\lambda/\partial x^i)_{x_0'}=0$. Let $\mathcal{V}'^1$ be the curve in M' defined by equations $x^2=\cdots=x^p=0$, $x^\lambda=H^\lambda(x^1,0,\cdots,0)$, $p+1\leqq\lambda\leqq n'$, and put $E'^k(x_0)=\rho_*(E^k(x_0))$. Clearly $\mathcal{V}'^1_{x_0}=E'^1(x_0)$. We want to show that there is an integral curve of Σ which covers $\mathcal{V}'^1$. Let $\bar\eta^1,\cdots,\bar\eta^{\alpha_1}$ be a basis of the space of forms of degree 1 of Σ' at the point x_0' and choose forms $\bar\xi^1,\cdots,\bar\xi^{\beta_1}$ such that $\bar\eta^1,\cdots,\bar\eta^{\alpha_1},\bar\xi^1,\cdots,\bar\xi^{\beta_1}$ is a basis of the forms of degree 1 of Σ at the point x_0 (we denote the form $\rho^*\bar\eta^i$ also by $\bar\eta^i$). Since x_0 is a regular point of Σ there are forms $\eta^1,\cdots,\eta^{\alpha_1},\xi^1,\cdots,\xi^{\beta_1}$ defined in a neighborhood of x_0 such that they are a basis of the space of 1-forms of Σ in this neighborhood and $\eta^i_{x_0}=\bar\eta^i$, $\xi^i_{x_0}=\bar\xi^i$. Put

$$(1)\qquad \eta_\beta^{\alpha}=\sum_{i=1}^{n'}A_i^{\alpha}(x)dx^i,\qquad 1\leqq\alpha\leqq\alpha_1,$$

$$(2)\qquad \xi^\beta=\sum_{i=1}^{n'}B_i^\beta(x,y)dx^i+\sum_{j=1}^{n}C_j^\beta(x,y)dy^j,\qquad 1\leqq\beta\leqq\beta_1.$$

From the hypothesis, the matrix $\|C_j^\beta(x,y)\|$ has maximum rank at the point x_0. Hence, the linear equations

$$(3)\qquad B_i^\beta(x,y)+\sum_{\lambda=p+1}^{n'}B_\lambda^\beta\frac{\partial H^\lambda}{\partial x^1}+\sum_{j=1}^{n}C_j^\beta(x,y)\frac{dy^j}{dx^1}=0$$

can be solved with respect to some of the variables dy^j/dx^i. Assume that they can be solved with respect to $dy^1/dx^1,\cdots,dy^s/dx^1$. In (3), put, $x^2=\cdots=x^{n'}=0$ and replace the variables $y^{s+1},\cdots,y^n$ by arbitrary functions $F^{s+1}(x^1),\cdots,F^n(x^1)$ such that $(dF^k/dx^1)=0$, $s+1\leqq k\leqq n$. Let $F^k(x^1)$, $1\leqq k\leqq s$, be the solution of the resulting system with the initial conditions $(dF^k/dx^1)_{x^1=0}=0$, $1\leqq h\leqq s$. Then the curve $x^2=\cdots=x^p=0$, $x^\lambda=H^\lambda(x^1,0,\cdots,0)$, $y^j=F^j(x^1)$, $p+1\leqq\lambda\leqq n'$, $1\leqq j\leqq n$, is an integral curve of Σ which covers $\mathcal{V}'^1$.

Assume now, by induction, that we have lifted the manifold $\mathcal{V}'^{r-1}$ defined by the equations $x^r=\cdots=x^p=0$, $x^\lambda=H^\lambda(x^1,\cdots,x^{r-1},0,\cdots,0)$ to an integral manifold $\mathcal{V}^{r-1}$ of Σ which is tangent $E^{r-1}(x_0)$. Let $\bar\eta^1,\cdots,\bar\eta^{\alpha_{r-1}},\bar\xi^1,\cdots,\bar\xi^{\beta_{r-1}}$ be a basis of $J(E^{r-1}(x_0))$ such that $\bar\eta^1,\cdots,\bar\eta^{\alpha_{r-1}}$, is a basis of $J'(E^{r-1}(x_0))$. For a contact element $E^r(x)$ sufficiently close to $E^r(x_0)$, we can write

$$dx^{\lambda} \mid E^{r}(x) = \sum_{i=1}^{r} u_i^{\lambda} dx^{i}, \qquad r+1 \leqq \lambda \leqq n',$$

$$dy^{\lambda} \mid E^{r}(x) = \sum_{i=1}^{r} v_i^{j} dx^{i}, \qquad 1 \leqq j \leqq n.$$

Put

$$L_i(E^r(x)) = \frac{\partial}{\partial x^i} + \sum_{\lambda=r+1}^{n'} u_i^{\lambda} \frac{\partial}{\partial x^{\lambda}} + \sum_{j=1}^{n} v_i^{j} \frac{\partial}{\partial y^j}, \qquad 1 \leqq i \leqq r,$$

and assume that

$$\bar{\eta}^{\alpha} = (L_{i_1}(E^r(x_0)) \wedge \cdots \wedge L_{i_{t-1}}(E^r(x_0)))\rfloor\omega, \qquad 1 \leqq \alpha \leqq \alpha_{r-1},$$

where ω is a form of degree t in $\rho^*\Sigma'$. For $E^r(x)$ sufficiently close to $E^r(x_0)$ define

$$\eta^{\alpha}(E^r(x)) = (L_{i_1}(E^r(x)) \wedge \cdots \wedge L_{i_{t-1}}(E^r(x)))\rfloor\omega.$$

Define forms ξ^{β} in a similar way. Let

$$\eta^{\alpha}(E^r(x)) = \sum_{i=1}^{n'} A_i^{\alpha}(x, u) dx^{i},$$

$$\xi^{\beta}(E^r(x)) = \sum_{i=1}^{n'} B_i^{\beta}(x, y, u, v) dx^{i} + \sum_{j=1}^{n} C_j^{\beta}(x, y, u, v) dy^{j},$$

be the expression of these forms in local coordinates. Observe that the coefficients are functions only of the variables x, y, u_i^{λ}, v_i^{j} with $1 \leqq i \leqq r-1$.

By hypothesis the matrix $\|C_j^{\beta}\|$ has maximum rank at the point $E^r(x_0)$. Assume that the equations of $\mathfrak{V}^{r-1}$ are $x^r = \cdots = x^p = 0$, $x^{\lambda} = H^{\lambda}(x^1, \cdots, x^{r-1}, 0, \cdots, 0)$, $y^j = F^j(x^1, \cdots, x^r)$ and construct the functions

$$\eta^{\alpha}(E^r(x))(L_r(E^r(x))) = A_r^{\alpha} + \sum_{\lambda=r+1}^{n'} A_{\lambda}^{\alpha} u_r^{\lambda},$$

$$\xi^{\beta}(E^r(x))(L_r(E\,(x))) = B_r^{\beta} + \sum_{\lambda=r+1}^{n'} B_{\lambda}^{\beta} u_r^{\lambda} + \sum_{j=1}^{n} C_j^{\beta} v_r^{j}.$$

Consider the following system of partial differential equations, obtained replacing in the above functions u_i^{λ} by $\partial x^{\lambda}/\partial x^i$ and v_i^j by $\partial y^j/\partial x^i$:

$$(4)\qquad A_r^\alpha\left(x, \frac{\partial x^\lambda}{\partial x^i}\right) + \sum_{\lambda=r+1}^{n'} A_\lambda^\alpha\left(x, \frac{\partial x^\lambda}{\partial x^i}\right)\frac{\partial x^\lambda}{\partial x^r} = 0,$$

$$(5)\qquad \begin{aligned} B_r^\beta\left(x, y, \frac{\partial x^\lambda}{\partial x^i}, \frac{y^j}{\partial x^i}\right) &+ \sum_{\lambda=r+1}^{n'} B_\lambda^\beta\left(x, y, \frac{\partial x^\lambda}{\partial x^i}, \frac{\partial y^j}{\partial x^i}\right)\frac{\partial x^\lambda}{\partial x^r} \\ &+ \sum_{j=1}^{n} C_j^\beta\left(x, y, \frac{\partial x^\lambda}{\partial x^i}, \frac{\partial y^j}{\partial x^i}\right)\frac{\partial y^j}{\partial x^r} = 0. \end{aligned}$$

We want to lift the manifold $\mathcal{V}'^r$ defined by $x^{r+1} = \cdots = x^p = 0$ $x^\lambda = H^\lambda(x^1, \cdots, x^r, 0, \cdots, 0)$ to an integral manifold of Σ. Observe that the functions $H^\lambda(x^1, \cdots, x^r, 0, \cdots, 0)$ are solutions of equations (4). Assume that the equations of $\mathcal{V}^{r-1}$ are $x^r = \cdots = x^p = 0$, $x^\lambda = H^\lambda(x^1, \cdots, x^{r-1}, 0, \cdots, 0)$, $y^j = G^j(x^1, \cdots, x^{r-1})$. Equations (5) can be solved with respect to some of the variables $\partial y^j/\partial x^r$; assume that they can be solved with respect to $\partial y^1/\partial x^r, \cdots, \partial y^s/\partial x^r$. Put in (5),

$$\frac{\partial x^{r+1}}{\partial x^r} = \cdots = \frac{\partial x^p}{\partial x^r} = 0, \qquad \frac{\partial x^\lambda}{\partial x^r} = \frac{\partial H^\lambda(x^1, \cdots, x^p)}{\partial x^r},$$

$$p + 1 \leqq \lambda \leqq n',$$

and replace the variables y^j by arbitrary functions $y^j = F^j(x^1, \cdots, x^r)$, $s+1 \leqq j \leqq n$, subjected to the restrictions $(\partial F^j/\partial x^i)_{(0, \ldots, 0)} = 0$ and $F^j(x^1, \cdots, x^{r-1}, 0) = G^j(x^1, \cdots, x^{r-1})$, $1 \leqq i \leqq r$, $s+1 \leqq j \leqq n$. Let $y^j = F^j(x^1, \cdots, x^r)$, $1 \leqq j \leqq s$, be the solution of the resulting Cauchy-Kowaleswky system with the initial conditions $F^j(x^1, \cdots, x^{r-1}, 0) = G^j(x^1, \cdots, x^{r-1})$, $1 \leqq j \leqq s$. Then, as in the Cartan-Kähler theorem, the manifold $\mathcal{V}^r$ defined by the equations $x^{r+1} = \cdots = x^p = 0$, $x^\lambda = H^\lambda(x^1, \cdots, x^r)$, $y^j = F^j(x^1, \cdots, x^r)$, $1 \leqq j \leqq n$, is a solution of Σ which covers $\mathcal{V}'^r$. The Theorem is proved.

References

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